

An approach toward Schubert positivities of polynomials using Kraśkiewicz-Pragacz modules

Masaki Watanabe

Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba Meguro-ku Tokyo 153-8914, Japan
mwata@ms.u-tokyo.ac.jp

Abstract

In this paper, we investigate properties of modules introduced by Kraśkiewicz and Pragacz which realize Schubert polynomials as their characters. In particular, we give some characterizations of modules having a filtration by Kraśkiewicz-Pragacz modules. In finding criteria for filtrations, we calculate generating sets for the annihilator ideals of the lowest vectors in Kraśkiewicz-Pragacz modules, and derive a projectivity result concerning Kraśkiewicz-Pragacz modules.

Keywords: Schubert polynomials, Schubert functors, Kraśkiewicz-Pragacz modules

1 Introduction

Though Schubert polynomials originally arose from the cohomology ring of flag varieties, they also have purely combinatorial interests apart from the geometry of flag varieties. Since Schubert polynomials are a kind of generalizations of Schur functions, it is an interesting problem to investigate analogues of several positivity properties of Schur functions for Schubert polynomials. For example, it is a classical result that $\mathfrak{S}_u \mathfrak{S}_v$ is a positive sum of Schubert polynomials, which is usually proved using the cohomology ring of flag varieties. Another such problem is a Schubert-positivity question for the “plethysm” of a Schur function with a Schubert polynomial. For a symmetric function s and a polynomial $f = x^\alpha + x^\beta + \cdots$, the plethysm of s and f is defined as $s[f] = s(x^\alpha, x^\beta, \dots)$ (cf. [10, §I.8]). The question is: is $s_\sigma[\mathfrak{S}_w]$ a positive sum of Schubert polynomials, for all partitions σ and permutations w ? In this paper, motivated by such positivity problems on Schubert polynomials, we provide some new results on the modules related with Schubert polynomials introduced by Kraśkiewicz and Pragacz ([5], [6]).

For a permutation w , Kraśkiewicz and Pragacz defined a certain representation \mathcal{S}_w of the Lie algebra \mathfrak{b} of all upper triangular matrices such that its character with respect to the subalgebra \mathfrak{h} of all diagonal matrices is equal to the Schubert polynomial \mathfrak{S}_w (precise definition of \mathcal{S}_w will be given in the sec-

tion 3). In this paper we call these modules *Kraśkiewicz-Pragacz modules* or *KP modules*.

Since the characters of KP modules are Schubert polynomials, the problems concerning Schubert positivity are deeply related to the class of modules having a filtration by KP modules. For instance, the Schubert positivity of $\mathfrak{S}_u \mathfrak{S}_v$ and $s_\sigma[\mathfrak{S}_w]$ will follow if one shows that $\mathcal{S}_u \otimes \mathcal{S}_v$ and $s_\sigma(\mathcal{S}_w)$ (here s_σ denote the Schur functor), respectively, have such filtrations.

KP modules are in some way similar to Demazure modules (of type A), the modules generated by an extremal vector in an irreducible representation of \mathfrak{gl}_n : they are both cyclic \mathfrak{b} -modules parametrized by the weight of the generators, and if the index permutation is 2143-avoiding then the KP module coincide with the Demazure modules with the same weight of the generator (note that in general they are different (see Example 3.5): if a permutation w does not avoid 2143 then there exists a *strict* surjection from \mathcal{S}_w to the Demazure module of corresponding lowest weight). In this paper, we develop an analog of the theory on Demazure modules ([4], [12], [14], [15, §3]) in the case of KP modules to obtain characterizations of modules having filtrations by KP modules.

The module \mathcal{S}_w is generated by its lowest weight vector u_w . In this paper we first show in Section 4 that the annihilator ideal $\text{Ann}_{\mathcal{U}(\mathfrak{n}^+)}(u_w)$, where \mathfrak{n}^+ is the Lie subalgebra of all strictly upper triangular matrices, is generated by the elements $e_{ij}^{m_{ij}(w)+1}$ ($1 \leq i < j \leq n$) for some integers $m_{ij}(w)$ which can be read off from w , where e_{ij} denotes the (i, j) -th matrix unit. This result can be seen as a generalization of a classical result which states that the finite dimensional irreducible representation of \mathfrak{gl}_n with lowest weight $-\lambda$ can be presented as $\mathcal{U}(\mathfrak{n}^+)/\langle e_i^{\langle \lambda, h_i \rangle + 1} \rangle_{1 \leq i \leq n-1}$ as a $\mathcal{U}(\mathfrak{n}^+)$ -module. This result can moreover be seen as an analog of the result on Demazure modules, given by Joseph ([4, Theorem 3.4]), which states, in the \mathfrak{gl}_n -case, that the annihilator of the generator of the Demazure module with lowest weight $\lambda \in \mathbb{Z}^n$ is generated by the elements $e_{ij}^{1+\max\{0, \lambda_j - \lambda_i\}}$ ($1 \leq i < j \leq n$).

Using this presentation of KP modules, in section 6 we characterize KP modules by their projectivity in certain categories; it is an analogue of Polo's theorem (originally for Demazure modules: see [12], [15, §3]) in the case of KP modules. Finally, using the results obtained so far, we obtain some criteria (Theorem 8.1, Theorem 8.2) for a module to have a filtration by KP modules, in a way similar to the argument given by van der Kallen ([14], [15, §3]) for Demazure modules using the method from the theory of highest-weight categories.

The paper is organized as follows. In Sections 2 and 3 we recall and define some basic notations and results about Schubert polynomials and KP modules. In Sections 4 and 5 we give a generating set for the annihilator ideal of the lowest weight vector in a KP module. In Section 6, we introduce a new ordering on the weight lattice and show some results relating KP modules with this ordering. In Sections 7 and 8, we obtain some characterizations of modules having a filtration by KP modules, using the results of the previous sections. Section 9 serves as a concluding remark by stating some future problems.

Acknowledgement. I would like to thank Katsuyuki Naoi for giving the author information on related materials.

2 Preliminaries

Let $\mathbb{Z}_{>0}$ be the set of all positive integers and let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. A *permutation* w is a bijection from $\mathbb{Z}_{>0}$ to itself which fixes all but finitely many points. Let S_∞ denote the group of all permutations. For a positive integer n , let $S_n = \{w \in S_\infty : w(i) = i \ (i > n)\}$ and $S_\infty^{(n)} = \{w \in S_\infty : w(n+1) < w(n+2) < \dots\}$. We sometimes write a permutation in its one-line form: i.e., write $[w(1)w(2)\dots]$ to mean $w \in S_\infty$. If $w \in S_n$, we may write $[w(1)w(2)\dots w(n)]$ to mean w . For $i < j$, let t_{ij} denote the permutation which exchanges i and j and fixes all other points. Let $s_i = t_{i,i+1}$. The *inversion diagram* of $w \in S_\infty$ is defined as $I(w) = \{(i, j) : i < j, w(i) > w(j)\}$. Let $\ell(w) = |I(w)|$ and $\text{sgn}(w) = (-1)^{\ell(w)}$. For $w \in S_\infty^{(n)}$, we define $\text{code}(w) = (\text{code}(w)_1, \dots, \text{code}(w)_n) \in \mathbb{Z}_{\geq 0}^n$ by $\text{code}(w)_i = \#\{j : i < j, w(i) > w(j)\}$: this is usually called the *Lehmer code* of w and it uniquely determines w . If $\lambda = \text{code}(w)$ we write $w = \text{perm}(\lambda)$.

For a polynomial $f = f(x_1, x_2, \dots)$ and $i \in \mathbb{Z}_{>0}$, we define $\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}$. For $w \in S_\infty$ we can assign its *Schubert polynomial* $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \dots]$, which is recursively defined by

- $\mathfrak{S}_w = x_1^{m-1} x_2^{m-2} \dots x_{m-1}$ if $w = w_0(m) = [m \ m-1 \ \dots \ 1]$ for some m , and
- $\mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w$ if $\ell(ws_i) < \ell(w)$.

We note the fact (see eg. [9]) that if $w \in S_n$ (resp. $S_\infty^{(n)}$) then \mathfrak{S}_w is a linear combination of $x_1^{a_1} \dots x_n^{a_n}$ with $a_i \in \{0, \dots, n-i\}$ (resp. a polynomial in x_1, \dots, x_n).

Schubert polynomials satisfy the following identity known as *transition*:

Proposition 2.1 ([9, (4.16)]). *Let $w \in S_\infty \setminus \{\text{id}\}$. Let $j \in \mathbb{Z}_{>0}$ be the maximal integer such that $w(j) > w(j+1)$ and take $k > j$ maximal with $w(j) > w(k)$. Let $v = wt_{jk}$. Let $i_1 < \dots < i_A$ be the all integers less than j such that $\ell(vt_{i_a j}) = \ell(v) + 1$, and let $w^{(a)} = vt_{i_a j}$. Then*

$$\mathfrak{S}_w = x_j \mathfrak{S}_v + \sum_{a=1}^A \mathfrak{S}_{w^{(a)}}.$$

Note that if $w \in S_\infty^{(n)}$, v and $w^{(1)}, \dots, w^{(A)}$ in the proposition above are also in $S_\infty^{(n)}$. Note also that $\text{code}(v) = \text{code}(w) - \epsilon_j$, where $\epsilon_j = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the j -th position. .

Hereafter in this paper, we fix a positive integer n . Let K be a field of characteristic zero. Let \mathfrak{b} be the Lie algebra of all $n \times n$ upper triangular K -matrices and let $\mathfrak{h} \subset \mathfrak{b}$ be the subalgebra of all diagonal matrices. Let $\mathcal{U}(\mathfrak{b})$ be the universal enveloping algebra of \mathfrak{b} . For a $\mathcal{U}(\mathfrak{b})$ -module M and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, let $M_\lambda = \{m \in M : hm = \langle \lambda, h \rangle m \ (\forall h \in \mathfrak{h})\}$ where $\langle \lambda, h \rangle = \sum \lambda_i h_i$. M_λ is called the *weight space of weight λ* or *λ -weight space*, and elements of M_λ are said to *have weight λ* . If $M_\lambda \neq 0$ then λ is said to be a *weight* of M . If M is the direct sum of its weight spaces and each weight space has finite dimension, then M is said to be a *weight module* and we define $\text{ch}(M) = \sum_\lambda \dim M_\lambda x^\lambda$ where $x^\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n}$. For $1 \leq i < j \leq n$, let $e_{ij} \in \mathfrak{b}$ be the

matrix with 1 at the (i, j) -position and all other coordinates 0. It is easy to see that if M is a $\mathcal{U}(\mathfrak{b})$ -module and $x \in M_\lambda$, then $e_{ij}x \in M_{\lambda+\epsilon_i-\epsilon_j}$,

For $\lambda \in \mathbb{Z}^n$, let K_λ denote the one-dimensional $\mathcal{U}(\mathfrak{b})$ -module where $h \in \mathfrak{h}$ acts by $\langle \lambda, h \rangle$ and e_{ij} acts by 0. Note that every finite-dimensional weight module admits a filtration by these one dimensional modules.

3 Kraśkiewicz-Pragacz modules

In [5] and [6], Kraśkiewicz and Pragacz defined certain $\mathcal{U}(\mathfrak{b})$ -modules which we call here *Kraśkiewicz-Pragacz modules* or *KP modules*. Here we use the following definition. Let $w \in S_\infty^{(n)}$. Let $K^n = \bigoplus_{1 \leq i \leq n} Ku_i$ be the vector representation of \mathfrak{b} . For each $j \in \mathbb{Z}_{>0}$, let $l_j = l_j(w) = \#\{i : (i, j) \in I(w)\}$, $\{i : (i, j) \in I(w)\} = \{i_1, \dots, i_{l_j}\}$ ($i_1 < \dots < i_{l_j}$), and $u_w^{(j)} = u_{i_1} \wedge \dots \wedge u_{i_{l_j}} \in \bigwedge^{l_j} K^n$. Note that $u_w^{(j)} \in \bigwedge^{l_j} K^{\min\{n, j-1\}}$. Let $u_w = u_w^{(1)} \otimes u_w^{(2)} \otimes \dots \in \bigwedge^{l_1} K^n \otimes \bigwedge^{l_2} K^n \otimes \dots$. Then the KP module \mathcal{S}_w associated to w is defined as $\mathcal{S}_w = \mathcal{U}(\mathfrak{b})u_w$.

Remark 3.1. It is also possible to define KP modules using so-called *Rothe diagram* $D(w) = \{(i, w(j)) : i < j, w(i) > w(j)\}$ of w instead of $I(w)$. Since $I(w)$ and $D(w)$ differ only by a rearrangement of columns it does not matter which to use. $D(w)$ has an advantage that it is easier to see with hand what the diagram looks like: drawing rays downward and to the right from the positions $(i, w(i))$ ($i = 1, 2, \dots$) and then the remaining boxes give $D(w)$ (see the figure below). Also, in [2] a basis for \mathcal{S}_w is constructed using certain labellings of Rothe diagram.

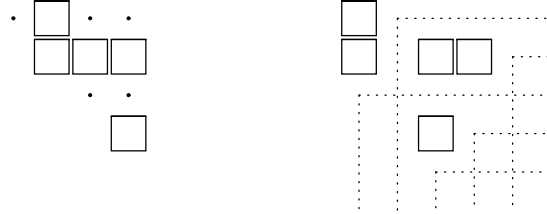


Figure 1: inversion diagram and Rothe diagram of the same permutation [25143].

KP modules have the following property:

Theorem 3.2 ([6, Remark 1.6 and Theorem 4.1]). \mathcal{S}_w is a weight module and $\text{ch}(\mathcal{S}_w) = \mathfrak{S}_w$.

Example 3.3. If $w = s_i$, then $I(s_i) = \{(i, i+1)\}$, $u_{s_i} = u_i$ and $\mathcal{S}_{s_i} = \bigoplus_{1 \leq j \leq i} Ku_j = K^i$. So $\text{ch}(\mathcal{S}_{s_i}) = x_1 + \dots + x_i = \mathfrak{S}_{s_i}$.

Example 3.4. More generally, if w is grassmannian, i.e. there exists a k such that $w(1) < \dots < w(k)$ and $w(k+1) < w(k+2) < \dots$, then the inversion diagram $I(w)$ of w is a “French-notation” Young diagram (see Figure 2). Thus in this case, u_w is a lowest-weight vector in certain irreducible representation of \mathfrak{gl}_k , and \mathcal{S}_w is equal to this representation (seen as a representation of \mathfrak{b}_n

through the morphism $\mathfrak{b}_n \ni e_{pq} \mapsto \begin{cases} e_{pq} & (q \leq k) \\ 0 & (q > k) \end{cases} \in \mathfrak{gl}_k$). This reflects the fact that the Schubert polynomial indexed by a grassmannian permutation is a Schur polynomial.

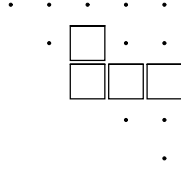


Figure 2: inversion diagram of a grassmannian permutation [136245] is a French-style Young diagram of shape (3, 1).

Example 3.5. More generally, if w is 2143-avoiding, then it can be seen that u_w is an extremal vector in an irreducible representation of \mathfrak{gl}_n (using the fact ([9, (1.27)]) that the rows of $I(w)$ for 2143-avoiding w is totally preordered by inclusion). Thus in this case the corresponding KP module \mathcal{S}_w is isomorphic to a Demazure module of \mathfrak{b} : i.e. a module generated by an extremal vector of an irreducible representation of \mathfrak{gl}_n . Note that this corresponds to the result of Lascoux and Schutzenberger ([8, Theorem 5], [7, Corollary 10.5.2]) that Schubert polynomials with 2143-avoiding indices are equal to certain key polynomials.

On the other hand, consider $w = [2143]$. Then $I(w) = \{(1, 2), (3, 4)\}$, $u_w = u_1 \otimes u_3$, $\mathcal{S}_w = \bigoplus_{1 \leq i \leq 3} K(u_1 \otimes u_i) = K^1 \otimes K^3$ and $\text{ch}(\mathcal{S}_w) = x_1(x_1 + x_2 + x_3) = \mathfrak{S}_w$. Note that in this case \mathcal{S}_w is *not* isomorphic to the Demazure module with the same lowest weight: \mathcal{S}_w is three-dimensional while the Demazure module with the same lowest weight is two-dimensional.¹ In general, \mathcal{S}_w is isomorphic to the Demazure module $V(\text{code}(w))$ with lowest weight $\text{code}(w)$ if and only if w is 2143-avoiding. We also note here that there always exists a surjection from \mathcal{S}_w to $V(\text{code}(w))$: this can be seen using the result from the next section and [4, Theorem 3.4].

In this paper we have to slightly extend the notion of Schubert polynomials and KP modules. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, we define the Schubert polynomial and the KP module associated to λ as follows. For $\lambda \in \mathbb{Z}_{\geq 0}^n$, let $\mathfrak{S}_\lambda = \mathfrak{S}_w$ and $\mathcal{S}_\lambda = \mathcal{S}_w$ where $w = \text{perm}(\lambda)$. For a general $\lambda \in \mathbb{Z}^n$, take $k \in \mathbb{Z}$ so that $\lambda + k\mathbf{1} \in \mathbb{Z}_{\geq 0}^n$, where $\mathbf{1} = (1, \dots, 1)$, and we define $\mathfrak{S}_\lambda = x^{-k\mathbf{1}}\mathfrak{S}_{\lambda+k\mathbf{1}}$ and $\mathcal{S}_\lambda = K_{-k\mathbf{1}} \otimes \mathcal{S}_{\lambda+k\mathbf{1}}$. Note that this definition does not depend on the choice of k , since if $\text{perm}(\lambda) = w$, then $\text{perm}(\lambda + \mathbf{1}) = \tilde{w} = [w(1) + 1 \cdots w(n) + 1 \ 1 \ w(n+1) + 1 \cdots]$, and $\mathfrak{S}_{\tilde{w}} = x^{\mathbf{1}}\mathfrak{S}_w$ and $\mathcal{S}_{\tilde{w}} = K_{\mathbf{1}} \otimes \mathcal{S}_w$ hold for them. It then follows from the theorem above that \mathcal{S}_λ is a weight module and $\text{ch}(\mathcal{S}_\lambda) = \mathfrak{S}_\lambda$ for all $\lambda \in \mathbb{Z}^n$. Note that, since \mathcal{S}_λ is generated by an element of weight λ , if $(\mathcal{S}_\lambda)_\mu \neq 0$ (i.e. if x^μ appears in \mathfrak{S}_λ with nonzero coefficient) then $\mu \succeq \lambda$, where \succeq denote the dominance order: $\mu \succeq \lambda$ iff $\mu - \lambda = \sum_{i=1}^{n-1} a_i(\epsilon_i - \epsilon_{i+1})$ for some $a_1, \dots, a_{n-1} \in \mathbb{Z}_{\geq 0}$. We also note here that for any $\mu, \nu \in \mathbb{Z}^n$, the number of $\lambda \in \mathbb{Z}^n$ with $\mu \succeq \lambda \succeq \nu$ is finite.

A *KP filtration* of a weight \mathfrak{b} -module M is a sequence $0 = M_0 \subset \cdots \subset M_r = M$ of weight \mathfrak{b} -modules such that each M_i/M_{i-1} is isomorphic to some KP module $\mathcal{S}_{\lambda^{(i)}}$. Note that if M has a KP filtration then $\text{ch}(M)$ is a positive sum of Schubert polynomials.

¹ The KP module $\mathcal{S}_{[2143]}$ in this example is, if not seen as a $\mathcal{U}(\mathfrak{b})$ -module but as a $\mathcal{U}(\mathfrak{n}^+)$ -module, isomorphic to a Demazure module (say $V(0, 0, 1)$); thus the results such as Theorem 4.1 for such kind of KP modules follow from known results on Demazure modules. But in fact there also exist KP modules which are, even as $\mathcal{U}(\mathfrak{n}^+)$ -modules, not isomorphic to any Demazure modules. An example is $\mathcal{S}_{[13254]} \cong K^2 \otimes K^4$.

4 Annihilator of the lowest weight vector

For $w \in S_\infty^{(n)}$ and $1 \leq i < j \leq n$, let $C_{ij}(w) = \{k : (i, k) \notin I(w), (j, k) \in I(w)\} = \{k : k > j, w(i) < w(k) < w(j)\}$ and let $m_{ij}(w) = |C_{ij}(w)| = \#\{k > j : w(i) < w(k) < w(j)\}$ (in particular, $m_{ij}(w) = 0$ if $w(i) > w(j)$). Since $e_{ij}^2 u_w^{(k)} = 0$ for $k \in C_{ij}(w)$ and $e_{ij} u_w^{(k)} = 0$ for $k \notin C_{ij}(w)$, we see that $e_{ij}^{m_{ij}(w)+1}$ annihilates $u_w = u_w^{(1)} \otimes u_w^{(2)} \otimes \cdots$. Let I_w denote the left ideal of $\mathcal{U}(\mathfrak{b})$ generated by $h - \langle \text{code}(w), h \rangle$ ($h \in \mathfrak{h}$) and $e_{ij}^{m_{ij}(w)+1}$ ($i < j$). Then, by the observation above and the fact that u_w has weight $\text{code}(w)$, there is a unique surjective morphism of $\mathcal{U}(\mathfrak{b})$ -modules from $\mathcal{U}(\mathfrak{b})/I_w$ to \mathcal{S}_w sending $1 \bmod I_w$ to u_w . We show the following:

Theorem 4.1. *The surjection above is an isomorphism.*

Remark 4.2. It is also possible to define u_D and \mathcal{S}_D for a general finite subset $D \subset \{1, \dots, n\} \times \mathbb{Z}_{>0}$ as in the same way we defined KP modules (\mathcal{S}_D is often called the *flagged Schur module* associated to D , see eg. [11, §7]; the equivalence of the definition there and our definition can be checked by the same argument as in [6, Remark 1.6]). Again in this setting, if we let $m_{ij}(D) = \#\{p : (i, p) \notin D, (j, p) \in D\}$ and $\lambda_i = \#\{p : (i, p) \in D\}$, then $e_{ij}^{m_{ij}(D)+1}$ ($i < j$) and $h - \langle \lambda, h \rangle$ ($h \in \mathfrak{h}$) annihilate u_D , and therefore we have a surjective morphism $\mathcal{U}(\mathfrak{b})/I_D \twoheadrightarrow \mathcal{S}_D$ where I_D is the left ideal generated by these elements. But this is not an isomorphism for general D : for example, if $D = \{(2, 1), (3, 2)\}$, then $\text{ch}(\mathcal{U}(\mathfrak{b})/I_D) = x_2 x_3 + x_1 x_3 + x_2^2 + 2x_1 x_2 + x_1^2 + x_1 x_2^2 x_3^{-1}$ while $\text{ch}(\mathcal{S}_D) = x_2 x_3 + x_1 x_3 + x_2^2 + 2x_1 x_2 + x_1^2$.

The theorem can be reduced to the following lemma, which will be proved in the next section:

Lemma 4.3. *Let $w \in S_\infty^{(n)} \setminus \{\text{id}\}$ and take j, i_1, \dots, i_A and $v, w^{(1)}, \dots, w^{(A)}$ as in Proposition 2.1. Let $x_a = e_{i_a j}^{m_{i_a j}(v)+1}$ for $a = 1, \dots, A$. Let $I^{(0)} = I_w$ and $I^{(a)} = I^{(a-1)} + \mathcal{U}(\mathfrak{b})x_a$ for $a = 1, \dots, A$. Also let I'_v be the left ideal of $\mathcal{U}(\mathfrak{b})$ generated by $h - \langle \text{code}(w), h \rangle = h - \langle \text{code}(v) + \epsilon_j, h \rangle$ ($h \in \mathfrak{h}$) and $e_{ij}^{m_{ij}(v)+1}$ ($i < j$), so $\mathcal{U}(\mathfrak{b})/I'_v \cong \mathcal{U}(\mathfrak{b})/I_v \otimes K_{\epsilon_j}$. Then $I'_v \subset I^{(A)}$ and $I_{w^{(a)}} x_a \subset I^{(a-1)}$ for $a = 1, \dots, A$.*

Here we show Theorem 4.1 assuming Lemma 4.3. Let $d_w = \dim \mathcal{U}(\mathfrak{b})/I_w$. The conclusion of Lemma 4.3 claims that there exist surjective morphisms $\mathcal{U}(\mathfrak{b})/I_v \otimes K_{\epsilon_j} \cong \mathcal{U}(\mathfrak{b})/I'_v \twoheadrightarrow \mathcal{U}(\mathfrak{b})/I^{(A)} : (x \bmod I'_v) \mapsto (x \bmod I^{(A)})$ and $\mathcal{U}(\mathfrak{b})/I_{w^{(a)}} \twoheadrightarrow I^{(a)}/I^{(a-1)} : (x \bmod I_{w^{(a)}}) \mapsto (xx_a \bmod I^{(a-1)})$ (note that $xx_a \in I^{(a)}$ since $x_a \in I^{(a)}$). Thus $\mathcal{U}(\mathfrak{b})/I_w = \mathcal{U}(\mathfrak{b})/I^{(0)}$ has a quotient filtration $\mathcal{U}(\mathfrak{b})/I^{(0)} \twoheadrightarrow \mathcal{U}(\mathfrak{b})/I^{(1)} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{U}(\mathfrak{b})/I^{(A)} \twoheadrightarrow 0$ with each subquotient being a quotient of $\mathcal{U}(\mathfrak{b})/I_{w^{(1)}}, \dots, \mathcal{U}(\mathfrak{b})/I_{w^{(A)}}$ and $\mathcal{U}(\mathfrak{b})/I_v \otimes K_{\epsilon_j}$ respectively. Therefore $d_w \leq d_{w^{(1)}} + \cdots + d_{w^{(A)}} + d_v$. So, by Proposition 2.1 and induction on lexicographic ordering on $(\ell(w), \mathfrak{S}_w(1))$, we see that $d_w \leq \mathfrak{S}_w(1)$ hold for any w . But on the other hand, we have a surjection $\mathcal{U}(\mathfrak{b})/I_w \twoheadrightarrow \mathcal{S}_w$ and thus $d_w \geq \dim \mathcal{S}_w = \mathfrak{S}_w(1)$. Thus $d_w = \mathfrak{S}_w(1)$ and the surjection above must be an isomorphism. This completes the proof of Theorem 4.1.

5 Proof of Lemma 4.3

Throughout this section, let $w \in S_\infty^{(n)} \setminus \{\text{id}\}$ and take $j, i_1, \dots, i_A, v, w^{(1)}, \dots, w^{(A)}$ as in Proposition 2.1. Take x_1, \dots, x_a and $I^{(0)}, \dots, I^{(A)}$ as in Lemma 4.3. Let $m_{pq} = m_{pq}(v)$ for $1 \leq p < q \leq n$. For $x, y, \dots, z \in \mathcal{U}(\mathfrak{b})$, let $\langle x, y, \dots, z \rangle$ denote the left ideal of $\mathcal{U}(\mathfrak{b})$ generated by x, y, \dots, z .

To make the calculations simple, we use the following basic fact from the representation theory of semisimple Lie algebras:

Proposition 5.1. *Let $\mathfrak{n}_3^+ = Ke_{12} \oplus Ke_{13} \oplus Ke_{23}$ be the Lie algebra of all 3×3 strictly upper triangular matrices which acts on $K^3 = Ku_1 \oplus Ku_2 \oplus Ku_3$ and $\bigwedge^2 K^3 = K(u_1 \wedge u_2) \oplus K(u_1 \wedge u_3) \oplus K(u_2 \wedge u_3)$ in the usual way. Then for $a, b \geq 0$, the $\mathcal{U}(\mathfrak{n}_3^+)$ -module generated by $(u_2 \wedge u_3)^a \otimes u_3^b \in S^a(\bigwedge^2 K^3) \otimes S^b(K^3)$ (S^\bullet denotes the symmetric product) is isomorphic to $\mathcal{U}(\mathfrak{n}_3^+)/I_{a,b}$ where $I_{a,b}$ is the left ideal of $\mathcal{U}(\mathfrak{n}_3^+)$ generated by e_{12}^{a+1} and e_{23}^{b+1} .*

Proof. First note that $(u_2 \wedge u_3)^a \otimes u_3^b$ is a lowest weight vector of an irreducible representation of \mathfrak{sl}_3 : i.e. $\mathcal{U}(\mathfrak{n}_3^+)((u_2 \wedge u_3)^a \otimes u_3^b)$ is an irreducible representation of \mathfrak{sl}_3 . Thus the claim is merely a well-known fact that a finite-dimensional irreducible representation $V(\lambda)$, with lowest weight λ , of a finite-dimensional semisimple Lie algebra \mathfrak{g} with simple root system Δ and upper-triangular part \mathfrak{n}^+ is isomorphic to $\mathcal{U}(\mathfrak{n}^+)/\langle e_\alpha^{\langle \lambda, h_\alpha \rangle} \rangle_{\alpha \in \Delta}$ as $\mathcal{U}(\mathfrak{n}^+)$ -modules ([3, Theorem 21.4]). \square

From this proposition, we have the following:

Lemma 5.2. *Let $f(x, y, z)$ be a polynomial (in non-commutative variables) and let $a, b \geq 0$. If $f(e_{12}, e_{13}, e_{23})((u_2 \wedge u_3)^a \otimes u_3^b) = 0$, then for $1 \leq p < q < r \leq n$, $f(e_{pq}, e_{pr}, e_{qr}) \in \langle e_{pq}^{a+1}, e_{qr}^{b+1} \rangle$.*

Proof. From Proposition 5.1 we have $f(e_{12}, e_{13}, e_{23}) \in \mathcal{U}(\mathfrak{n}_3^+)e_{12}^{a+1} + \mathcal{U}(\mathfrak{n}_3^+)e_{23}^{b+1}$, i.e. $f(e_{12}, e_{13}, e_{23}) = g(e_{12}, e_{13}, e_{23})e_{12}^{a+1} + h(e_{12}, e_{13}, e_{23})e_{23}^{b+1}$ for some g and h . Then $f(e_{pq}, e_{pr}, e_{qr}) = g(e_{pq}, e_{pr}, e_{qr})e_{pq}^{a+1} + h(e_{pq}, e_{pr}, e_{qr})e_{qr}^{b+1} \in \langle e_{pq}^{a+1}, e_{qr}^{b+1} \rangle$. \square

With this lemma in hand, it is easy to prove the following:

Lemma 5.3. *For $1 \leq p < q < r \leq n$ and $N, M, N', M' \geq 0$,*

- (1) $e_{pr}^N e_{qr}^M \equiv 0 \pmod{\langle e_{pq}^{N'+1}, e_{qr}^{M'+1} \rangle}$ if $N + M > N' + M'$.
- (2) $e_{pq}^N e_{pr}^M \equiv 0 \pmod{\langle e_{pq}^{N'+1}, e_{qr}^{M'+1} \rangle}$ if $N + M > N' + M'$.
- (3) $e_{pr}^N \equiv \frac{(-1)^N}{N!} e_{qr}^N e_{pq}^N \pmod{\langle e_{pq}^{M+1}, e_{qr} \rangle}$ (and in fact mod $\langle e_{qr} \rangle$, although we do not need it here).
- (4) $e_{pr}^N \equiv \frac{1}{N!} e_{pq}^N e_{qr}^N \pmod{\langle e_{pq}, e_{qr}^{M+1} \rangle}$ (and mod $\langle e_{pq} \rangle$: we do not need it here).
- (5) $e_{pq}^{N+M+1} e_{qr}^M \equiv 0 \pmod{\langle e_{pq}^{N+1}, e_{qr}^{M+1} \rangle}$.
- (6) $e_{pq}^N e_{qr}^M \equiv 0 \pmod{\langle e_{pq}, e_{pr}^N, e_{qr}^{M+1} \rangle}$.

Proof. (1)-(5) follows from straightforward calculations checking the condition of Lemma 5.2. (6) also follows from Lemma 5.2, since $e_{12}^N e_{23}^M u_3^M = (\text{const.}) \cdot u_1^N u_2^{M-N} = (\text{const.}) \cdot e_{23}^{M-N} e_{13}^N u_3^M$ so $e_{pq}^N e_{qr}^M - (\text{const.}) \cdot e_{qr}^{M-N} e_{pr}^N \in \langle e_{pq}, e_{qr}^{M+1} \rangle$. \square

Let us move on to the proof of Lemma 4.3. First we prove $I'_v \subset I^{(A)}$. Since $h - \langle \text{code}(w), h \rangle \in I_w \subset I^{(A)}$, it suffices to show $e_{pq}^{m_{pq}+1} \in I^{(A)}$ for all $1 \leq p < q \leq n$. If $q \neq j$, we have $m_{pq} = m_{pq}(w)$ so $e_{pq}^{m_{pq}+1} \in I_w \subset I^{(A)}$. If $q = j$ and $v(p) > v(j)$, then $m_{pq} = 0 = m_{pq}(w)$ (note that, by the choice of k , there does not exist $r > j$ such that $w(k) < w(r) < w(j)$), and thus again $e_{pq}^{m_{pq}+1} \in I_w \subset I^{(A)}$. If $q = j$ and $p = i_a$, we have $e_{i_a j}^{m_{i_a j}+1} = x_a \in I^{(a)} \subset I^{(A)}$. Otherwise (i.e. if $q = j$, $v(p) < v(j)$ and $p \neq i_1, \dots, i_A$), the conclusion follows from the following lemma:

Lemma 5.4. *Let $p < j$, $v(p) < v(j)$ and $p \neq i_1, \dots, i_A$. Then*

- (1) *There exists some $a \in \{1, \dots, A\}$ such that $v(i_a) > v(p)$.*
- (2) *Let $a \in \{1, \dots, A\}$ be the maximal index such that $v(i_a) > v(p)$. Then $e_{pj}^{m_{pj}+1} \in I^{(a)}$.*

Proof. (1): By the assumptions we have $\ell(vt_{pj}) > \ell(v) + 1$, and thus there exists an i such that $p < i < j$ and $v(p) < v(i) < v(j)$. Take i to be maximal among such. Then there does not exist i' such that $i < i' < j$ and $v(i) < v(i') < v(j)$, and thus $\ell(vt_{ij}) = \ell(v) + 1$. Therefore i is in $\{i_1, \dots, i_A\}$. This shows (1) since $v(i) > v(p)$.

(2): Let $i = i_a$. First we claim that there exists no r such that $i < r < j$ and $v(p) < v(r) < v(i)$. Suppose such r exists. Take r to be maximal among such. Then by the same argument as in (1) we see that r is in $\{i_1, \dots, i_A\}$, and since $i < r$ we have $r = i_b$ for some $b > a$. This contradicts to the choice of a .

From the claim we see $m_{pi} = \#\{r > i : v(p) < v(r) < v(i)\} = \#\{r > j : v(p) < v(r) < v(i)\} = m_{pj} - m_{ij}$. So by Lemma 5.3(1), $e_{pj}^{m_{pj}+1} \in \langle e_{pi}^{m_{pi}+1}, e_{ij}^{m_{ij}+1} \rangle$. Since $e_{pi}^{m_{pi}+1} \in I_w \subset I^{(a)}$ and $e_{ij}^{m_{ij}+1} = x_a \in I^{(a)}$ we are done. \square

Let us now prove $I_{w^{(a)}} x_a \subset I^{(a-1)}$ ($a = 1, \dots, A$). Fix $a \in \{1, \dots, A\}$ and let $i = i_a$. We want to prove $(h - \langle \text{code}(w^{(a)}), h \rangle) x_a \in I^{(a-1)}$ for all $h \in \mathfrak{h}$ and $e_{pq}^{m_{pq}(w^{(a)})+1} x_a \in I^{(a-1)}$ for all $p < q$. We first check $(h - \langle \text{code}(w^{(a)}), h \rangle) x_a \in I^{(a-1)}$, i.e., the element $x_a \bmod I^{(a-1)}$ has weight $\text{code}(w^{(a)})$. It is easy to see that $\text{code}(w^{(a)}) = \text{code}(w) + (m_{ij} + 1)\epsilon_i - m_{ij}\epsilon_j = \text{code}(w) + (m_{ij} + 1)(\epsilon_i - \epsilon_j)$. On the other hand, $x_a \bmod I^{(a-1)} = e_{ij}^{m_{ij}+1} \bmod I^{(a-1)}$ has weight $\text{code}(w) + (m_{ij} + 1)(\epsilon_i - \epsilon_j)$ since $1 \bmod I^{(a-1)}$ has weight $\text{code}(w)$ and e_{ij} shifts the weight by $\epsilon_i - \epsilon_j$. This shows the claim.

We now check $e_{pq}^{m_{pq}(w^{(a)})+1} x_a = e_{pq}^{m_{pq}(w^{(a)})+1} e_{ij}^{m_{ij}+1}$ is in $I^{(a-1)}$ for all $1 \leq p < q \leq n$, case by case. First note that, by Lemma 5.4 and the consideration before that lemma, $e_{pq}^{m_{pq}+1} \in I^{(a-1)}$ unless $q = j$ and $v(p) \leq v(i)$, and in such case we see $e_{pq}^{m_{pq}+2} = e_{pq}^{m_{pq}(w)+1} \in I_w \subset I^{(a-1)}$. Also note that there does not exist an r such that $i < r < j$ and $v(i) < v(r) < v(j)$, since $\ell(vt_{ij}) = \ell(v) + 1$.

- $q > j$: In this case we have $m_{pq}(w^{(a)}) = 0 = m_{pq}(w)$, since both w and $w^{(a)}$ are increasing from $(j+1)$ -th position and thus there are no $r > q$ with $w(r) < w(q)$ or $w^{(a)}(r) < w^{(a)}(q)$. If $p \neq j$, $e_{pq}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{pq} \in I^{(a-1)}$ since $e_{pq} \in I^{(a-1)}$. If $p = j$, $e_{jq}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{jq} - (m_{ij}+1)e_{ij}^{m_{ij}}e_{iq} \in I^{(a-1)}$ since $e_{jq}, e_{iq} \in I^{(a-1)}$.
- $p = i$ and $q = j$: Trivial from $m_{ij}(w^{(a)}) = 0$ and $e_{ij}^{m_{ij}(w^{(a)})+1}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+2} \in I^{(a-1)}$.

Hereafter we assume $p < q \leq j$ and $(p, q) \neq (i, j)$.

- $\{p, q\} \cap \{i, j\} = \emptyset$: If $m_{pq}(w^{(a)}) = m_{pq}$ the proof is trivial since in this case $e_{pq}^{m_{pq}(w^{(a)})+1} \in I^{(a-1)}$ and $e_{pq}^{m_{pq}(w^{(a)})+1}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{pq}^{m_{pq}(w^{(a)})+1}$.

Consider the case $m_{pq}(w^{(a)}) \neq m_{pq}$. Then:

- $v(p) < v(q)$ must hold since otherwise $m_{pq}(w^{(a)}) = 0 = m_{pq}$,
- q must be larger than i , since otherwise $\{w^{(a)}(r) : r > q, w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(q)\} = \{v(r) : r > q, v(p) < v(r) < v(q)\}$ because $w^{(a)}$ and v only differ at i -th and j -th positions, and
- exactly one of $v(i)$ and $v(j)$ must lie between $v(p)$ and $v(q)$ since otherwise $\{r > q : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(q)\} = \{r > q : v(p) < v(r) < v(q)\}$.

Since $i < q < j$ and $\ell(vt_{ij}) = \ell(v) + 1$, the case $v(p) < v(i) < v(q) < v(j)$ cannot occur. So $v(i) < v(p) < v(j) < v(q)$. Then we have $p < i$ by the same reason. So we have $p < i < q < j$ and $v(i) < v(p) < v(j) < v(q)$.

Here $m_{pq}(w^{(a)}) = m_{pq} - 1$. Using the fact that there exists no $i < r < j$ with $v(i) < v(r) < v(j)$, we obtain $m_{iq} - m_{ij} = \#\{r > q : v(j) \leq v(r) < v(q)\} = m_{pq} - m_{pj}$.

We have $e_{pq}^{m_{pq}}e_{ij}^{m_{ij}+1} \equiv \frac{(-1)^{m_{ij}+1}}{(m_{ij}+1)!}e_{pq}^{m_{pq}}e_{qj}^{m_{ij}+1}e_{iq}^{m_{ij}+1} \pmod{I^{(a-1)}}$ by Lemma 5.3(3) since $e_{qj}, e_{iq}^{m_{iq}+1} \in I^{(a-1)}$. Using $[e_{pq}, e_{qj}] = e_{pj}$ and $[e_{pq}, e_{pj}] = [e_{qj}, e_{pj}] = 0$ we see that the RHS is a linear combination of $e_{qj}^{m_{ij}+1-\nu}e_{pq}^{m_{pq}-\nu}e_{pj}^{\nu}e_{iq}^{m_{ij}+1}$ ($\nu \geq 0$). Thus it suffices to show that these elements are in $I^{(a-1)}$ for each ν . If $\nu > m_{pj}$ it is clear since $[e_{pj}, e_{iq}] = 0$ and $e_{pj}^{m_{pj}+1} \in I^{(a-1)}$. Otherwise, it suffices to show $e_{pq}^{m_{pq}-\nu}e_{iq}^{m_{ij}+1} \in I^{(a-1)}$ since $[e_{pq}, e_{pj}] = 0$. This follows from $e_{pq}^{m_{pq}-m_{pj}}e_{iq}^{m_{ij}+1} = e_{pq}^{m_{iq}-m_{ij}}e_{iq}^{m_{ij}+1} \in I^{(a-1)}$, which can be deduced from $e_{pi}, e_{iq}^{m_{iq}+1} \in I^{(a-1)}$ using Lemma 5.3(1).

- $p = i$: Since $i < q < j$, the case $v(i) < v(q) < v(j)$ cannot occur. If $v(q) < v(i)$, we have $w^{(a)}(q) < w^{(a)}(i)$ and thus $m_{iq}(w^{(a)}) = 0$. Therefore $e_{iq}^{m_{iq}(w^{(a)})+1}e_{ij}^{m_{ij}+1} = e_{iq}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{iq} \in I^{(a-1)}$ since $e_{iq} \in I^{(a-1)}$. If $v(q) > v(j)$, $m_{iq}(w^{(a)}) = m_{iq} - m_{ij} - 1$ since $\{r > q : w^{(a)}(i) < w^{(a)}(r) < w^{(a)}(q)\} = \{r > q : v(i) < v(r) < v(q)\} \setminus (\{r > q : v(i) < v(r) < v(j)\} \cup \{j\}) = \{r > q : v(i) < v(r) < v(q)\} \setminus (\{r > j : v(i) < v(r) < v(j)\} \cup \{j\})$,

so we want to show $e_{iq}^{m_{iq}-m_{ij}} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. This follows from Lemma 5.3(2) since $e_{iq}^{m_{iq}+1}, e_{qj} \in I^{(a-1)}$.

- $q = i$: Here we have three cases to consider. If $v(p) < v(i)$, we have $m_{pi}(w^{(a)}) = m_{pi} + m_{ij} + 1$ since $\{r > i : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(i)\} = \{r > i : v(p) < v(r) < v(i)\} \cup \{r > i : v(i) < v(r) < v(j)\} \cup \{j\} = \{r > i : v(p) < v(r) < v(i)\} \cup \{r > j : v(i) < v(r) < v(j)\} \cup \{j\}$, and so we want to show $e_{pi}^{m_{pi}+m_{ij}+2} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. This follows from Lemma 5.3(5) since $e_{pi}^{m_{pi}+1}, e_{ij}^{m_{ij}+2} \in I^{(a-1)}$. If $v(i) < v(p) < v(j)$, we have $m_{pi}(w^{(a)}) = m_{pj}$ since $\{r > i : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(i)\} = \{r > i : v(p) < v(r) < v(j)\} = \{r > j : v(p) < v(r) < v(j)\}$ and so we want to show $e_{pi}^{m_{pj}+1} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. This follows from Lemma 5.3(6) since $e_{pi}, e_{ij}^{m_{ij}+2}, e_{pj}^{m_{pj}+1} \in I^{(a-1)}$. Finally if $v(p) > v(j)$, we have $w^{(a)}(p) > w^{(a)}(i)$, $m_{pi}(w^{(a)}) = 0$ and so we want to show $e_{pi} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. This follows from $e_{pi} e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1} e_{pi} + (m_{ij}+1) e_{ij}^{m_{ij}} e_{pj}$ and $e_{pi}, e_{pj} \in I^{(a-1)}$.
- $q = j$: This case consists of four subcases:
 - $p < i$ and $v(p) < v(i)$: Here $m_{pj}(w^{(a)}) = m_{pj} - m_{ij}$ since $\{r > j : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(j)\} = \{r > j : v(p) < v(r) < v(j)\} \setminus \{r > j : v(i) < v(r) < v(j)\}$. So we want to show $e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. If there is no r such that $i < r < j$ and $v(p) < v(r) < v(i)$, then $m_{pi} = m_{pj} - m_{ij}$, and thus $e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} = e_{pi}^{m_{pi}+1} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ by Lemma 5.3(1) since $e_{pi}^{m_{pi}+1}, e_{ij}^{m_{ij}+2} \in I^{(a-1)}$. If there exists such r , take r to be the largest among such ones. Then $m_{pr} = m_{pj} - m_{rj}$, since there exists no s such that $r < s < j$ and $v(p) < v(s) < v(r)$. By $e_{ir}, e_{rj}^{m_{rj}+2} \in I^{(a-1)}$ and Lemma 5.3(4), we have $e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} \equiv \frac{1}{(m_{ij}+1)!} e_{pj}^{m_{pj}-m_{ij}+1} e_{ir}^{m_{ij}+1} e_{rj}^{m_{ij}+1} \pmod{I^{(a-1)}}$. Since the elements $e_{pr}^{m_{pr}+1} = e_{pr}^{m_{pj}-m_{rj}+1}$ and $e_{rj}^{m_{rj}+2}$ are in $I^{(a-1)}$ we see from Lemma 5.3(1) that $e_{pj}^{m_{pj}-m_{ij}+1} e_{rj}^{m_{ij}+1} \in I^{(a-1)}$. Thus $e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} e_{rj}^{m_{ij}+1} = e_{ij}^{m_{ij}+1} e_{pj}^{m_{pj}-m_{ij}+1} e_{rj}^{m_{ij}+1} \in I^{(a-1)}$ and this shows the claim.
 - $p < i$ and $v(p) > v(i)$: Here $m_{pj}(w^{(a)}) = 0$ since $w^{(a)}(p) > w^{(a)}(j)$. Thus $e_{pj}^{m_{pj}(w^{(a)})+1} e_{ij}^{m_{ij}+1} = e_{pj} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ by $e_{pi}, e_{ij}^{m_{ij}+2} \in I^{(a-1)}$ and Lemma 5.3(1).
 - $p > i$ and $v(p) < v(i)$: Here $m_{pj}(w^{(a)}) = m_{pj} - m_{ij}$ since $\{r > j : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(j)\} = \{r > j : v(p) < v(r) < v(j)\} \setminus \{r > j : v(i) < v(r) < v(j)\}$. Thus $e_{pj}^{m_{pj}(w^{(a)})+1} e_{ij}^{m_{ij}+1} = e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ by $e_{ip}, e_{pj}^{m_{pj}+2} \in I^{(a-1)}$ and Lemma 5.3(1).
 - $p > i$ and $v(p) > v(j)$: Here $m_{pj}(w^{(a)}) = 0$ since $w^{(a)}(p) > w^{(a)}(j)$. Thus $e_{pj}^{m_{pj}(w^{(a)})+1} e_{ij}^{m_{ij}+1} = e_{pj} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ since $e_{pj} e_{ij}^{m_{ij}+2} = e_{ij}^{m_{ij}+2} e_{pj}$ and $e_{pj} \in I^{(a-1)}$.

Thus we checked $e_{pq}^{m_{pq}(w^{(a)})+1} x_a \in I^{(a-1)}$ for all $p < q$. This finishes the proof of Lemma 4.3. \square

Remark 5.5. It is clear from the definition that $m_{pr}(w) \leq m_{pq}(w) + m_{qr}(w)$ for any $p < q < r$. If $m_{pr}(w) = m_{pq}(w) + m_{qr}(w)$, then by Lemma 5.3(1) we have $e_{pr}^{m_{pr}(w)+1} \in \langle e_{pq}^{m_{pq}(w)+1}, e_{qr}^{m_{qr}(w)+1} \rangle$. Thus in fact the generators $e_{pr}^{m_{pr}(w)+1}$ such that there exists some $q \in \{p+1, \dots, r-1\}$ with $m_{pr}(w) = m_{pq}(w) + m_{qr}(w)$ are superfluous.

6 Projectivity of KP modules

In this section we characterize KP modules by their projectivities in certain categories. This can be seen as an analog of Polo's theorem ([12, Corollary 2.5], [15, Theorem 3.1.10]) for the case of KP modules.

Let \mathcal{C} be the category of all weight modules. For $\Lambda \subset \mathbb{Z}^n$, let \mathcal{C}_Λ be the full subcategory of \mathcal{C} consists of all weight modules whose weights are in Λ . Note that if $|\Lambda| < \infty$ and $\Lambda' = \{\rho - \lambda : \lambda \in \Lambda\}$ ($\rho = (n-1, n-2, \dots, 0)$), then $\mathcal{C}_{\Lambda'} \cong \mathcal{C}_\Lambda^{\text{op}}$ by $M \mapsto M^* \otimes K_\rho$ (it is also true for infinite Λ if we take M^* to be the graded dual $\bigoplus (M_\lambda)^*$ of M).

Lemma 6.1 (cf. [15, Lemma 3.1.1]). *For finite $\Lambda \subset \mathbb{Z}^n$, \mathcal{C}_Λ has enough projectives (it is also true for infinite Λ if we allow the weight spaces of a weight module to be infinite dimensional).*

Proof. For $\lambda \in \Lambda$, let $P_\lambda = \mathcal{U}(\mathfrak{b}) / \langle h - \langle h, \lambda \rangle \rangle_{h \in \mathfrak{h}}$ (which is isomorphic to $\mathcal{U}(\mathfrak{n}^+)$ as a $\mathcal{U}(\mathfrak{n}^+)$ -module, by PBW theorem) and let P_λ^Λ be the largest quotient of P_λ which is in \mathcal{C}_Λ , i.e. P_λ^Λ is the quotient of P_λ by the submodule generated by all weight spaces $(P_\lambda)_\mu$ ($\mu \notin \Lambda$). Then P_λ^Λ is projective in \mathcal{C}_Λ since for $N \in \mathcal{C}_\Lambda$, $\text{Hom}(P_\lambda^\Lambda, N) = \text{Hom}(P_\lambda, N) = N_\lambda$.

For general $M \in \mathcal{C}_\Lambda$, $P_M = \bigoplus_\lambda (P_\lambda^\Lambda)^{\oplus \dim M_\lambda}$ is a projective object in \mathcal{C}_Λ and there is a surjection $P_M \twoheadrightarrow M$. This shows the lemma. \square

Note that, if $\lambda \in \Lambda$, P_λ^Λ has the maximum proper submodule $\bigoplus_{\mu \neq \lambda} (P_\lambda^\Lambda)_\mu$; therefore the head of P_λ^Λ is K_λ , and thus P_λ^Λ is the projective cover of K_λ in \mathcal{C}_Λ .

We introduce two orderings (other than dominance order) on \mathbb{Z}^n as follows. For two permutations $w, v \in S_\infty$, we write $w \leq_{\text{lex}} v$ if $w = v$ or there exists an $i \in \mathbb{Z}_{>0}$ such that $w(j) = v(j)$ for all $j < i$ and $w(i) < v(i)$. Likewise, we write $w \leq_{\text{rlex}} v$ if $w = v$ or there exists an $i \in \mathbb{Z}_{>0}$ such that $w(j) = v(j)$ for all $j > i$ and $w(i) < v(i)$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, define $|\lambda| = \sum \lambda_i$. If $\lambda, \mu \in \mathbb{Z}_{\geq 0}^n$ and $w = \text{perm}(\lambda), v = \text{perm}(\mu)$, we write $\lambda \geq_{\text{lex}} \mu$ if $|\lambda| = |\mu|$ and $w^{-1} \leq_{\text{lex}} v^{-1}$. For general λ and μ in \mathbb{Z}^n , take k so that $\lambda + k\mathbf{1}$ and $\mu + k\mathbf{1}$ are in $\mathbb{Z}_{\geq 0}^n$, and define $\lambda \geq_{\text{lex}} \mu \iff \lambda + k\mathbf{1} \geq_{\text{lex}} \mu + k\mathbf{1}$. Note that this definition does not depend on the choice of k since $\text{perm}(\lambda)^{-1} \leq_{\text{lex}} \text{perm}(\mu)^{-1} \iff \text{perm}(\lambda + \mathbf{1})^{-1} \leq_{\text{lex}} \text{perm}(\mu + \mathbf{1})^{-1}$ for $\lambda, \mu \in \mathbb{Z}_{\geq 0}^n$. We define the other ordering \geq' in the same way, except that we use \leq_{rlex} instead of \leq_{lex} . We prepare the following two lemmas about these orderings:

Lemma 6.2. *For $\lambda, \mu \in \mathbb{Z}^n$, $\lambda \geq \mu$ if and only if $\rho - \lambda \geq' \rho - \mu$.*

Proof. We may assume $|\lambda| = |\mu|$. We only need to prove the “only if” direction since the other implication follows by exchanging λ and μ . Take integers L and M so that $\lambda + L\mathbf{1}, \mu + L\mathbf{1}, \rho - \lambda + M\mathbf{1}, \rho - \mu + M\mathbf{1} \in \mathbb{Z}_{\geq 0}^n$. Let $w = \text{perm}(\lambda + L\mathbf{1}), v = \text{perm}(\mu + L\mathbf{1}), w' = \text{perm}(\rho - \lambda + M\mathbf{1})$ and $v' = \text{perm}(\rho - \mu + M\mathbf{1})$. Then $w, v, w', v' \in S_{\infty}^{(n)} \cap S_N$, and these permutations are related by $w'(i) = N + 1 - w(i), v'(i) = N + 1 - v(i)$ for $i = 1, \dots, n$, where $N = n + L + M$. Thus, for $p \in \{1, \dots, N\}$, $w'^{-1}(p) \leq n$ (resp. $v'^{-1}(p) \leq n$) if and only if $w^{-1}(N + 1 - p) \leq n$ (resp. $v^{-1}(N + 1 - p) \leq n$), and $w'^{-1}(p) = \begin{cases} w^{-1}(N + 1 - p) & (w^{-1}(p) \leq n) \\ n + N + 1 - w^{-1}(N + 1 - p) & (w^{-1}(p) > n) \end{cases}$ and $v'^{-1}(p) = \begin{cases} v^{-1}(N + 1 - p) & (v^{-1}(p) \leq n) \\ n + N + 1 - v^{-1}(N + 1 - p) & (v^{-1}(p) > n) \end{cases}$.

Now assume $w \leq_{\text{lex}} v$. if $w = v$ we have nothing to prove so we assume that there is an i such that $w^{-1}(1) = v^{-1}(1), \dots, w^{-1}(i-1) = v^{-1}(i-1), w^{-1}(i) < v^{-1}(i)$. By the above description of w' and v' it is clear that $w'^{-1}(j) = v'^{-1}(j)$ for $j > N + 1 - i$. We show $w'^{-1}(N + 1 - i) < v'^{-1}(N + 1 - i)$. If $w^{-1}(i) < v^{-1}(i) \leq n$ we have $w'^{-1}(N + 1 - i) = w^{-1}(i) < v^{-1}(i) = v'^{-1}(N + 1 - i)$. If $w^{-1}(1) \leq n < v^{-1}(i)$ we have $w'^{-1}(N + 1 - i) \leq n < v'^{-1}(N + 1 - i)$. The case $n < w^{-1}(i) < v^{-1}(i)$ cannot occur, since in such case $w^{-1}(i) = n + 1 + \#\{j < i : w^{-1}(j) > n\}$, $v^{-1}(i) = n + 1 + \#\{j < i : v^{-1}(j) > n\}$ and $\{j < i : w^{-1}(j) > n\} = \{j < i : v^{-1}(j) > n\}$. Thus we have checked $w'^{-1}(N + 1 - i) < v'^{-1}(N + 1 - i)$ and thus $w'^{-1} \leq_{\text{rlex}} v'^{-1}$. This shows the lemma. \square

Lemma 6.3. *For any $\lambda \in \mathbb{Z}^n$, the set $\{\nu : \nu \leq \lambda\}$ is finite and linearly ordered by \leq .*

Proof. Linear-orderedness is clear from the definition of \leq . We claim that if $\lambda, \mu \in \mathbb{Z}^n$, $|\lambda| = |\mu|$ and $\min_i \lambda_i > \min_i \mu_i$ then $\lambda < \mu$ (this shows the lemma since there exists only finitely many $\nu \in \mathbb{Z}^n$ such that $|\nu| = |\lambda|$ and $\min_i \nu_i \geq \min_i \lambda_i$). We may assume that $\lambda, \mu \in \mathbb{Z}_{\geq 0}^n$. Let $m = \min_i \mu_i$. Then $w = \text{perm}(\lambda)$ and $v = \text{perm}(\mu)$ satisfy $w^{-1}(1) = v^{-1}(1) = n + 1, \dots, w^{-1}(m) = v^{-1}(m) = n + m$ and $w^{-1}(m + 1) > n \geq v^{-1}(m + 1)$. Thus $w^{-1} \geq_{\text{lex}} v^{-1}$. \square

We define $\mathcal{C}_{\leq \lambda} = \mathcal{C}_{\{\nu : \nu \leq \lambda\}}$, $\mathcal{C}_{< \lambda} = \mathcal{C}_{\{\nu : \nu < \lambda\}}$ and $\mathcal{C}_{\leq' \lambda} = \mathcal{C}_{\{\nu : \nu \leq' \lambda\}}$. The main result of this section is the following proposition:

Proposition 6.4. *For $\lambda \in \mathbb{Z}^n$, the modules \mathcal{S}_{λ} and $\mathcal{S}_{\rho-\lambda}^* \otimes K_{\rho}$ are in $\mathcal{C}_{\leq \lambda}$. Moreover, \mathcal{S}_{λ} is projective and $\mathcal{S}_{\rho-\lambda}^* \otimes K_{\rho}$ is injective in $\mathcal{C}_{\leq \lambda}$.*

Note that, by the remark before Lemma 6.1, the last claim is equivalent to the claim that $\mathcal{S}_{\rho-\lambda}$ is projective in $\mathcal{C}_{\{\rho-\nu : \nu \leq \lambda\}} = \mathcal{C}_{\leq' \rho-\lambda}$. Moreover, since the head of \mathcal{S}_{λ} is K_{λ} , this proposition claims that \mathcal{S}_{λ} is the projective cover of K_{λ} in both $\mathcal{C}_{\leq \lambda}$ and $\mathcal{C}_{\leq' \lambda}$, i.e., $\mathcal{S}_{\lambda} \cong P_{\lambda}^{\leq \lambda} \cong P_{\lambda}^{\leq' \lambda}$ (we write $P_{\mu}^{\leq \lambda}$ and $P_{\mu}^{\leq' \lambda}$ for $P_{\mu}^{\{\nu : \nu \leq \lambda\}}$ and $P_{\mu}^{\{\nu : \nu \leq' \lambda\}}$ respectively). We also remark that the proposition implies the same statement for \leq' instead of \leq , by Lemma 6.2.

To prove Proposition 6.4, we have to prove the following four facts: for every $\lambda, \mu \in \mathbb{Z}^n$,

- (1) $(\mathcal{S}_\lambda)_\mu \neq 0$ implies $\lambda \geq \mu$,
- (2) $(\mathcal{S}_{\rho-\lambda}^* \otimes K_\rho)_\mu \neq 0$ (which is equivalent to $(\mathcal{S}_{\rho-\lambda})_{\rho-\mu} \neq 0$) implies $\lambda \geq \mu$,
- (3) $\text{Ext}^1(\mathcal{S}_\lambda, K_\mu) \neq 0$ implies $\lambda < \mu$ (here Ext^1 is taken in either \mathcal{C} or $\mathcal{C}_{\leq \lambda}$, which does not matter since $\mathcal{C}_{\leq \lambda}$ is closed under extension), and
- (4) $\text{Ext}^1(\mathcal{S}_{\rho-\lambda}, K_{\rho-\mu}) \neq 0$ implies $\lambda < \mu$.

Before starting the proof, first let us make a observation on the weights of \mathcal{S}_w ($w \in S_\infty^{(n)}$). Let $l_j(w) = \#\{i : i < j, w(i) > w(j)\}$ as in the definition of KP modules. Since \mathcal{S}_w is a submodule of $\bigotimes_{j \geq 1} \bigwedge^{l_j(w)} K^{j-1}$, any weight of \mathcal{S}_w is a weight of $\bigotimes_{j \geq 1} \bigwedge^{l_j(w)} K^{j-1}$. The weights of the latter space can be understood as follows. A *w-pattern* (terminology only for here) is a sequence of sets (I_1, I_2, \dots) such that $I_j \subset \{1, \dots, j-1\}$ and $|I_j| = l_j(w)$. Define the *weight* (μ_1, μ_2, \dots) of a *w-pattern* (I_1, I_2, \dots) by $\mu_i = \#\{j : i \in I_j\}$. Then it is easy to see that μ is a weight of $\bigotimes_{j \geq 1} \bigwedge^{l_j(w)} K^{j-1}$ if and only if it is the weight of some *w-pattern*.

Let us now prove (1) and (2) above.

(1): We may assume that λ and μ are in $\mathbb{Z}_{\geq 0}^n$, since $(\mathcal{S}_\lambda)_\mu \neq 0 \iff (\mathcal{S}_{\lambda+k\mathbf{1}})_{\mu+k\mathbf{1}} \neq 0$ for any $\lambda, \mu \in \mathbb{Z}^n$ and any $k \in \mathbb{Z}$. Let $w = \text{perm}(\lambda)$ and $v = \text{perm}(\mu)$. We prove a stronger statement: if μ is the weight of some *w-pattern* (I_1, I_2, \dots) then $\lambda \geq \mu$.

We first show $w^{-1}(1) \leq v^{-1}(1)$. Let $i = w^{-1}(1)$. Since $w(1), \dots, w(i-1) > w(i)$ we have $l_i(w) = i-1$, and thus $I_i = \{1, \dots, i-1\}$. Thus $\mu_1, \dots, \mu_{i-1} \geq 1$. Since $v^{-1}(1) = \min\{j : \mu_j = 0\}$, this shows $w^{-1}(1) \leq v^{-1}(1)$.

Now consider the case $w^{-1}(1) = v^{-1}(1)$. In this case we have $\mu_i = 0$, i.e. none of the sets I_j contains i . Define $\sigma_i : \mathbb{Z}_{>0} \setminus \{i\} \rightarrow \mathbb{Z}_{>0}$ by $\sigma_i(i') =$

$$\begin{cases} i' & (i' < i) \\ i' - 1 & (i' > i) \end{cases}, \text{ and consider a new sequence of sets } I' = (\sigma_i(I_1), \dots, \sigma_i(I_{i-1}), \sigma_i(I_{i+1}), \sigma_i(I_{i+2}), \dots).$$

It is easy to check that I' is a w' -pattern with weight code (v') , where $w' = [w(1)-1 \ \dots \ w(i-1)-1 \ w(i+1)-1 \ w(i+2)-1 \ \dots]$ and $v' = [v(1)-1 \ \dots \ v(i-1)-1 \ v(i+1)-1 \ v(i+2)-1 \ \dots]$. An inductive argument shows that $w'^{-1} \leq_{\text{lex}} v'^{-1}$. This shows $w^{-1} \leq_{\text{lex}} v^{-1}$. \square

(2): We may assume $\lambda, \mu \in \mathbb{Z}_{\geq 0}^n$ as before. Let $w = \text{perm}(\lambda)$ and $v = \text{perm}(\mu)$. We prove a stronger statement: if μ is the weight of some *w-pattern* (I_1, I_2, \dots) then $\rho - \lambda \geq \rho - \mu$, or equivalently (by Lemma 6.2), $\lambda \geq' \mu$. Take N so that $w, v \in S_N$. Note that $I_{N+1} = I_{N+2} = \dots = \emptyset$ since $l_w(N+1) = l_w(N+2) = \dots = 0$.

We first show $w^{-1}(N) \leq v^{-1}(N)$. Let $i = w^{-1}(N)$. Then we have $l_i(w) = 0$ and thus $I_i = \emptyset$. Thus for $j < i$, we have $j \notin I_1, \dots, I_j$ and $j \notin I_i$. Thus $\mu_j \leq N - j - 1$. Since $v^{-1}(N) = \min\{i : \mu_i = N - i\}$ this shows $v^{-1}(N) \geq w^{-1}(N)$.

Now consider the case $w^{-1}(N) = v^{-1}(N)$. Then $\mu_i = N - i$. Since $i \notin I_1, \dots, I_i$ we must have $i \in I_{i+1}, \dots, I_N$. It is easy to see that $I' = (\sigma_i(I_1), \dots, \sigma_i(I_{i-1}), \sigma_i(I_{i+1} \setminus \{i\}), \dots, \sigma_i(I_N \setminus \{i\}), \emptyset, \emptyset, \dots)$ is a w' -pattern with weight code (v') where $w' = [w(1) \ \dots \ w(i-1) \ w(i+1) \ \dots \ w(N)]$ and $v' = [v(1) \ \dots \ v(i-1) \ v(i+1) \ \dots \ v(N)]$. An inductive argument shows $w'^{-1} \leq_{\text{rlex}} v'^{-1}$.

This shows $w^{-1} \leq_{\text{rlex}} v^{-1}$. \square

For (3) and (4), we need the following observation. By Theorem 4.1, for any $w \in S_\infty^{(n)}$ there is a projective resolution of \mathcal{S}_w in \mathcal{C} of the form $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathcal{S}_w \rightarrow 0$ with $P_0 = P_{\text{code}(w)}$ and $P_1 = \bigoplus_{p < q} P_{\text{code}(w) + (m_{pq}(w) + 1)(\epsilon_p - \epsilon_q)}$. Here by Remark 5.5, we can in fact replace P_1 by a smaller module: sum over all $p < q$ such that

(*) : there does not exist $p < r < q$ with $m_{pq}(w) = m_{pr}(w) + m_{rq}(w)$.

In particular, $\text{Ext}^1(\mathcal{S}_w, K_\mu) = 0$ unless $\mu = \text{code}(w) + (m_{pq}(w) + 1)(\epsilon_p - \epsilon_q)$ for some $p < q$ satisfying the property (*) above.

(3): We may assume that $\lambda, \mu \in \mathbb{Z}_{\geq 0}^n$, since $\lambda < \mu \iff \lambda + k\mathbf{1} < \mu + k\mathbf{1}$ and $\text{Ext}^1(\mathcal{S}_\lambda, K_\mu) \neq 0 \iff \text{Ext}^1(\mathcal{S}_{\lambda+k\mathbf{1}}, K_{\mu+k\mathbf{1}}) \neq 0$ for any $\lambda, \mu \in \mathbb{Z}^n$ and any $k \in \mathbb{Z}$.

Let $w = \text{perm}(\lambda)$ and $v = \text{perm}(\mu)$. By the remark above, we have $\mu = \lambda + (m_{pq}(w) + 1)(\epsilon_p - \epsilon_q)$ for some $p < q$ (and therefore $w \neq v$). We first show $w^{-1}(1) \geq v^{-1}(1)$. Let $i = w^{-1}(1)$. If $i < v^{-1}(1)$, then $\mu_i > 0$ while $\lambda_i = 0$, and so $p = i$. But then $m_{pq}(w) = \text{code}(w)_q = \lambda_q$ and so we have $\mu_q = -1$, which contradicts to $\mu \in \mathbb{Z}_{\geq 0}^n$. Therefore $i \geq v^{-1}(1)$.

If $i = v^{-1}(1)$, then $\lambda_i = \mu_i = 0$, and so $p, q \neq i$. Therefore $\lambda' = (\lambda_1 - 1, \dots, \lambda_{i-1} - 1, \lambda_{i+1}, \lambda_{i+2}, \dots)$ and $\mu' = (\mu_1 - 1, \dots, \mu_{i-1} - 1, \mu_{i+1}, \mu_{i+2}, \dots)$ satisfy $\mu' = \lambda' + (m_{pq}(w) + 1)(\epsilon_{p'} - \epsilon_{q'})$ for $p' = \sigma_i(p)$, $q' = \sigma_i(q)$. Moreover, $m_{pq}(w) = m_{p'q'}(w')$, where $w' = [w(1) - 1 \cdots w(i-1) - 1 \ w(i+1) - 1 \ w(i+2) - 1 \cdots]$. Thus an inductive argument shows $w'^{-1} \geq_{\text{lex}} v'^{-1}$ where $v' = [v(1) - 1 \cdots v(i-1) - 1 \ v(i+1) - 1 \ v(i+2) - 1 \cdots]$. This shows $w^{-1} \geq_{\text{lex}} v^{-1}$. \square

(4): We assume $\text{Ext}^1(\mathcal{S}_\lambda, K_\mu) \neq 0$ and prove $\rho - \lambda < \rho - \mu$, or equivalently, $\lambda < \mu$. We may assume that $\lambda, \mu \in \mathbb{Z}_{\geq 0}^n$ as before. Let $w = \text{perm}(\lambda)$, $v = \text{perm}(\mu)$. Take N so that $w, v \in S_N$. We have $\mu = \lambda + (m_{pq}(w) + 1)(\epsilon_p - \epsilon_q)$ for some $p < q$ as before, with the property (*) remarked above. We first show $w^{-1}(N) \geq v^{-1}(N)$.

Assume $w^{-1}(N) < v^{-1}(N)$. Then $\lambda_{w^{-1}(N)} = N - w^{-1}(N)$ while $\mu_{w^{-1}(N)} < N - w^{-1}(N)$ and so $q = w^{-1}(N)$.

We first claim that there does not exist r such that $p < r < q$ and $w(p) < w(r)$. Suppose such r exists. Take r to be the largest among such. By the property (*) we have $m_{pq}(w) < m_{pr}(w) + m_{rq}(w)$. This means that there is a column index $1 \leq j \leq N$ such that $(p, j), (q, j) \in I(w)$, $(r, j) \notin I(w)$ or $(p, j), (q, j) \notin I(w)$, $(r, j) \in I(w)$, since other types of column contribute to LHS and RHS by the same value. We see that neither of these cases cannot occur as follows.

- Assume the former case. Then $(p, j) \in I(w)$ implies $w(j) < w(p) < w(r)$ and $(q, j) \in I(w)$ implies $j > q > r$. These shows $(r, j) \in I(w)$. Contradiction.
- Assume the latter case. $w(q) = N > w(j)$ and $(q, j) \notin I(w)$ implies $j < q$. Also, $(r, j) \in I(w)$ implies $j > r > p$, and this together with $(p, j) \notin I(w)$ shows $w(p) < w(j)$. Thus j satisfies $p < j < q$, $w(p) < w(j)$ and $j > r$. This contradicts to the choice of r .

Since there does not exist r such that $p < r < q$ and $w(p) < w(r)$, we see that $m_{pq}(w) = \#\{r > q : w(p) < w(r) < w(q)\} = \#\{r > q : w(p) < w(r)\} =$

$N - w(p) - 1 - \#\{r < q : w(p) < w(r)\} = N - w(p) - 1 - \#\{r < p : w(p) < w(r)\}$. From this and $\lambda_p = \text{code}(w)_p = \#\{r > p : w(r) < w(p)\} = w(p) - 1 - \#\{r < p : w(r) < w(p)\}$, we see $\mu_p = \lambda_p + m_{pq}(w) + 1 = N - p$. This means $v^{-1}(N) = \min\{p' : \mu_{p'} = N - p'\} \leq p < q = w^{-1}(N)$. This contradicts to the assumption and thus we see $w^{-1}(N) \geq v^{-1}(N)$.

If $w^{-1}(N) = v^{-1}(N)$, then $p \neq w^{-1}(N) \neq q$ as before, and we can inductively argue in the same way as in (3). \square

7 Vanishing of higher extensions

In this section, we prove an analogue of “Strong form of Polo’s theorem” ([15, Theorem 3.2.2]) for KP modules: i.e. the vanishing of higher extensions $\text{Ext}^i(\mathcal{S}_\lambda, \mathcal{S}_\mu^* \otimes K_\rho)$ ($i \geq 1$). To prove it we need the following lemma:

Lemma 7.1. *For any $\lambda \geq \lambda'$, $M, N \in \mathcal{C}_{\leq \lambda'}$ and $i \geq 0$, $\text{Ext}_{\leq \lambda}^i(M, N) \cong \text{Ext}_{\leq \lambda'}^i(M, N)$. Here $\text{Ext}_{\leq \lambda}^i$ is short for $\text{Ext}_{\mathcal{C}_{\leq \lambda}}^i$.*

Proof. It is enough (by Lemma 6.3) to prove $\text{Ext}_{\leq \lambda}^i(M, N) = \text{Ext}_{\mathcal{C}_{< \lambda}}^i(M, N)$ for $M, N \in \mathcal{C}_{< \lambda}$. Take a projective resolution $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ such that each P_i is a direct sum of some modules $P_\mu^{\leq \lambda}$ with $\mu \leq \lambda$ (in fact, the only indecomposable projectives in $\mathcal{C}_{\leq \lambda}$ are $P_\mu^{\leq \lambda}$, so this condition is superfluous). For $L \in \mathcal{C}_{\leq \lambda}$, let \overline{L} be the largest quotient of L which is in $\mathcal{C}_{< \lambda}$, i.e., \overline{L} is the quotient of L by the submodule generated by the weight space L_λ of weight λ . Note that if $P_i = P_\mu^{\leq \lambda} \oplus P_\nu^{\leq \lambda} \oplus \cdots$, then $\overline{P_i} = P_\mu^{< \lambda} \oplus P_\nu^{< \lambda} \oplus \cdots$ where $P_\mu^{< \lambda}, P_\nu^{< \lambda}, \dots$ are the largest quotients of P_μ, P_ν, \dots which are in $\mathcal{C}_{< \lambda}$. We are done if we show that $\cdots \rightarrow \overline{P_1} \rightarrow \overline{P_0} \rightarrow M \rightarrow 0$ is a projective resolution of M , since $\text{Hom}(\overline{P_i}, N) = \text{Hom}(P_i, N)$. It is clear that each $\overline{P_i}$ is projective. Let Ker_i be the kernel of $P_i \rightarrow \overline{P_i}$. Since $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ is exact, the exactness of $\cdots \rightarrow \overline{P_1} \rightarrow \overline{P_0} \rightarrow M \rightarrow 0$ is equivalent to that of $\cdots \rightarrow \text{Ker}_1 \rightarrow \text{Ker}_0 \rightarrow 0$.

For any $\mu, \nu \leq \lambda$, we have a linear map $(P_\mu^{\leq \lambda})_\lambda \otimes (P_\nu^{\leq \lambda})_\nu \rightarrow (P_\mu^{\leq \lambda})_\nu$ defined by $xu_\mu \otimes yu_\nu \mapsto yxu_\mu$ for $x \in \mathcal{U}(\mathfrak{n}^+)_{\lambda-\mu}$ and $y \in \mathcal{U}(\mathfrak{n}^+)_{\nu-\lambda}$ where u_μ is the image of $1 \otimes 1 \in \mathcal{U}(\mathfrak{n}^+) \otimes K_\mu \cong P_\mu \rightarrow P_\mu^{\leq \lambda}$ (this definition does not depend on the choice of y since the submodule of $P_\mu^{\leq \lambda}$ generated by xu_μ is a quotient of $P_\lambda^{\leq \lambda}$ by definition). This map induces a surjection from $(P_\mu^{\leq \lambda})_\lambda \otimes (P_\nu^{\leq \lambda})_\nu$ to $\text{Ker}((P_\mu^{\leq \lambda})_\nu \rightarrow (P_\nu^{\leq \lambda})_\nu)$, since the kernel is, by definition, generated by $(P_\mu^{\leq \lambda})_\lambda$ as a $\mathcal{U}(\mathfrak{b})$ -module. We claim that this surjection is in fact an isomorphism, for any λ and any $\mu, \nu \leq \lambda$. Note that the claim implies the lemma: if we show this we have $(P_i)_\lambda \otimes (P_\lambda^{\leq \lambda})_\nu \cong (\text{Ker}_i)_\nu$ for each i , and thus the exactness of $\cdots \rightarrow \text{Ker}_1 \rightarrow \text{Ker}_0 \rightarrow 0$ follows from that of $\cdots \rightarrow (P_1)_\lambda \rightarrow (P_0)_\lambda \rightarrow 0$.

For $\lambda \in \mathbb{Z}^n$ and $\mu, \nu \leq \lambda$, we have a quotient filtration $(P_\mu^{\leq \lambda})_\nu = (P_\mu^{\leq \kappa^{(r)}})_\nu \rightarrow (P_\mu^{\leq \kappa^{(r-1)}})_\nu \rightarrow \cdots \rightarrow (P_\mu^{\leq \kappa^{(1)}})_\nu \rightarrow 0$, where $\kappa^{(1)} < \cdots < \kappa^{(r)} = \lambda$ are the elements of \mathbb{Z}^n less than or equal to λ . By the argument above, the subquotient $\text{Ker}((P_\mu^{\leq \kappa^{(i)}})_\nu \rightarrow (P_\mu^{\leq \kappa^{(i-1)}})_\nu)$ of this filtration is a quotient of $(P_\mu^{\leq \kappa^{(i)}})_{\kappa^{(i)}} \otimes (P_{\kappa^{(i)}}^{\leq \kappa^{(i)}})_\nu$. Thus $\dim(P_\mu^{\leq \lambda})_\nu \leq \sum_{\kappa \leq \lambda} \dim((P_\mu^{\leq \kappa})_\kappa \otimes (P_\kappa^{\leq \kappa})_\nu)$. If we show that the equality holds, then the desired isomorphism $(P_\mu^{\leq \kappa^{(i)}})_{\kappa^{(i)}} \otimes (P_{\kappa^{(i)}}^{\leq \kappa^{(i)}})_\nu \cong \text{Ker}((P_\mu^{\leq \kappa^{(i)}})_\nu \rightarrow (P_\mu^{\leq \kappa^{(i-1)}})_\nu)$ follows for all i : in particular, proving the equality

for a sufficiently large λ (with respect to the ordering \leq) is enough for the proof of Lemma 7.1.

We know $(P_\kappa^{\leq \kappa})_\nu \cong (\mathcal{S}_\kappa)_\nu$ by Proposition 6.4. Now consider $(P_\mu^{\leq \kappa})_\kappa$. Since $P_\mu^{\leq \kappa}$ is the quotient of P_μ by the submodule generated by all weight spaces $(P_\mu)_\sigma$ ($\sigma \not\leq \kappa$), we see that

$$(P_\mu^{\leq \kappa})_\kappa \cong \mathcal{U}(\mathfrak{n}^+)_{\kappa-\mu} / \text{Span}_K\{xy : x \in \mathcal{U}(\mathfrak{n}^+)_{\kappa-\sigma}, y \in \mathcal{U}(\mathfrak{n}^+)_{\sigma-\mu} \text{ for some } \sigma \not\leq \kappa\}.$$

The algebra antiautomorphism on $\mathcal{U}(\mathfrak{n}^+)$ given by $X \mapsto X$ ($X \in \mathfrak{n}^+$) induces an isomorphism between this space and

$$\begin{aligned} & \mathcal{U}(\mathfrak{n}^+)_{\kappa-\mu} / \text{Span}_K\{yx : x \in \mathcal{U}(\mathfrak{n}^+)_{\kappa-\sigma}, y \in \mathcal{U}(\mathfrak{n}^+)_{\sigma-\mu} \text{ for some } \sigma \not\leq \kappa\} \\ &= \mathcal{U}(\mathfrak{n}^+)_{\kappa-\mu} / \text{Span}_K\{yx : x \in \mathcal{U}(\mathfrak{n}^+)_{\kappa-\sigma}, y \in \mathcal{U}(\mathfrak{n}^+)_{\sigma-\mu} \text{ for some } \sigma \text{ s.t. } \rho - \sigma \not\leq' \rho - \kappa\}. \end{aligned}$$

By the same argument as above we see that this is isomorphic to $(P_{\rho-\kappa}^{\leq' \rho-\kappa})_{\rho-\mu}$. By Proposition 6.4 (and Lemma 6.2) we see $(P_{\rho-\kappa}^{\leq' \rho-\kappa})_{\rho-\mu} \cong (\mathcal{S}_{\rho-\kappa})_{\rho-\mu}$. Thus, after all, we see that $(P_\mu^{\leq \kappa})_\kappa \cong (\mathcal{S}_{\rho-\kappa})_{\rho-\mu}$.

Since as we have seen above $(P_\mu^{\leq \kappa})_\kappa \cong (\mathcal{S}_{\rho-\kappa})_{\rho-\mu}$ and $(P_\kappa^{\leq \kappa})_\nu \cong (\mathcal{S}_\kappa)_\nu$, we see that $\dim((P_\mu^{\leq \kappa})_\kappa \otimes (P_\kappa^{\leq \kappa})_\nu)$ is equal to the coefficient of $x^{\rho-\mu}y^\nu$ in $\mathfrak{S}_{\rho-\kappa}(x)\mathfrak{S}_\kappa(y)$. Also, $\dim(P_\mu^{\leq \lambda})_\nu = \dim \mathcal{U}(\mathfrak{n}^+)_{\nu-\mu}$ if λ is sufficiently large with respect to \leq . Thus the proof of Lemma 7.1 is now reduced to the following elementary lemma:

Lemma 7.2. *For $\mu, \nu \in \mathbb{Z}^n$, $\dim \mathcal{U}(\mathfrak{n}^+)_{\nu-\mu}$ is equal to the coefficient of $x^{\rho-\mu}y^\nu$ in $\sum_{\kappa \in \mathbb{Z}^n} \mathfrak{S}_{\rho-\kappa}(x)\mathfrak{S}_\kappa(y)$.*

Let us prove this lemma. We use the following result from [13]:

Lemma 7.3 ([13, Lemma 6.2 and Corollary 9.2, reformulated]). *For a positive integer N , define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$ by $\langle x^\alpha, x^\beta \rangle = \delta_{\alpha\beta}$. Then for $w, v \in S_N$, $\langle \mathfrak{S}_w, \mathfrak{S}_{w_0v}(x_1^{-1}, \dots, x_N^{-1}) \prod_{1 \leq i < j \leq N} (x_i - x_j) \rangle = \delta_{wv}$, where $w_0 = [N \ N-1 \ \dots \ 1] \in S_N$.*

We slightly modify this lemma into a form which is more suitable for our use:

Lemma 7.4. *If we define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by $\langle x^\alpha, x^\beta \rangle = \delta_{\alpha\beta}$, then for $\lambda, \mu \in \mathbb{Z}^n$, $\langle \mathfrak{S}_\lambda, \mathfrak{S}_{\rho-\mu}(x_1^{-1}, \dots, x_n^{-1}) \prod_{1 \leq i < j \leq n} (x_i - x_j) \rangle = \delta_{\lambda\mu}$.*

Proof. We may assume that $\lambda, \mu \in \mathbb{Z}_{\geq 0}^n$. Let $w = \text{perm}(\lambda), v = \text{perm}(\mu)$. Take N so that $w, v \in S_N$. Then by the previous lemma, we have

$$\langle \mathfrak{S}_w, \mathfrak{S}_{w_0v}(x_1^{-1}, \dots, x_N^{-1}) \prod_{1 \leq i < j \leq N} (x_i - x_j) \rangle = \delta_{wv} = \delta_{\lambda\mu} \quad \dots (*)$$

where $w_0 = [N \ N-1 \ \dots \ 1] \in S_N$.

Since $\prod_{1 \leq i < j \leq N} (x_i - x_j) = \prod_{i \leq n < j} (x_i - x_j) \cdot \prod_{i < j \leq n} (x_i - x_j) \cdot \prod_{n < i < j} (x_i - x_j)$, it can be seen that

$$\begin{aligned} \prod_{1 \leq i < j \leq N} (x_i - x_j) &\equiv (x_1 \cdots x_n)^{N-n} \cdot \prod_{i < j \leq n} (x_i - x_j) \cdot \prod_{n < i < j} (x_i - x_j) \\ &= (x_1 \cdots x_n)^{N-n} \cdot \prod_{i < j \leq n} (x_i - x_j) \cdot (x_{n+1}^{N-n-1} x_{n+2}^{N-n-2} \cdots x_{N-1} + R) \end{aligned}$$

modulo terms whose total degree in variables x_{n+1}, \dots, x_N is strictly larger than $T = \binom{N-n}{2}$, and R is some polynomial in x_{n+1}, \dots, x_N with degree T and without monomial $x_{n+1}^{N-n-1} x_{n+2}^{N-n-2} \dots x_{N-1}$.

Let f be the sum of all terms in $\mathfrak{S}_{w_0 v}$ whose degree in x_{n+1}, \dots, x_N is equal to T . Note that, since $\mathfrak{S}_{w_0 v}$ is a linear combination of monomials $x_1^{a_1} \dots x_n^{a_n}$ ($0 \leq a_i \leq N-i$), the degree in x_{n+1}, \dots, x_N of its terms are always at most T : that is, $\mathfrak{S}_{w_0 v} = f + (\text{terms with degree} < T \text{ in variables } x_{n+1}, \dots, x_N)$. Also note $f \in x_{n+1}^{N-n-1} \dots x_{N-1} \mathbb{Z}[x_1, \dots, x_n]$ by the same reason. We claim $f = (x_1 \dots x_n)^{N-n} x_{n+1}^{N-n-1} \dots x_{N-1} \mathfrak{S}_{\rho-\mu}$.

Let $w_{n,N} = [1 \dots n \ N \ N-1 \dots n+1] \in S_N$. Note that $\text{code}(w_{n,N} w_0 v) = \rho - \mu + (N-n)\mathbf{1}$ and thus $\mathfrak{S}_{w_{n,N} w_0 v} = (x_1 \dots x_n)^{N-n} \mathfrak{S}_{\rho-\mu}$. We have $\mathfrak{S}_{w_{n,N} w_0 v} = \partial_{w_{n,N}} \mathfrak{S}_{w_0 v}$, where $\partial_{w_{n,N}} = (\partial_{n+1} \partial_{n+2} \dots \partial_{N-1}) \cdot (\partial_{n+2} \dots \partial_{N-1}) \dots \partial_{N-1}$. Since the operators ∂_i ($n+1 \leq i \leq N-1$) lower the degree in variables x_{n+1}, \dots, x_N by one, $\partial_{w_{n,N}}$ annihilates $\mathfrak{S}_{w_0 v} - f$. Thus $\mathfrak{S}_{w_{n,N} w_0 v} = \partial_{w_{n,N}} f$. Since $f \in x_{n+1}^{N-n-1} \dots x_{N-1} \mathbb{Z}[x_1, \dots, x_n]$ and $\partial_{w_{n,N}} x_{n+1}^{N-n-1} \dots x_{N-1} = 1$ we see that $\partial_{w_{n,N}} f = f / (x_{n+1}^{N-n-1} \dots x_{N-1})$. Thus $f = x_{n+1}^{N-n-1} \dots x_{N-1} \mathfrak{S}_{w_{n,N} w_0 v} = (x_1 \dots x_n)^{N-n} x_{n+1}^{N-n-1} \dots x_{N-1} \mathfrak{S}_{\rho-\mu}$. This shows the claim above.

We have seen that

$$\prod_{1 \leq i < j \leq N} (x_i - x_j) \equiv (x_1 \dots x_n)^{N-n} \cdot \prod_{i < j \leq n} (x_i - x_j) \cdot (x_{n+1}^{N-n-1} x_{n+2}^{N-n-2} \dots x_{N-1} + R)$$

and

$$\mathfrak{S}_{w_0 v}(x_1^{-1}, \dots, x_N^{-1}) \equiv (x_1 \dots x_n)^{-N+n} x_{n+1}^{-N+n+1} \dots x_{N-1}^{-1} \cdot \mathfrak{S}_{\rho-\mu}(x_1^{-1}, \dots, x_n^{-1})$$

modulo terms having degrees $> T$ and $> -T$ in variables x_{n+1}, \dots, x_N respectively. Thus $\mathfrak{S}_{w_0 v}(x_1^{-1}, \dots, x_N^{-1}) \prod_{1 \leq i < j \leq N} (x_i - x_j)$ is equal to

$$\mathfrak{S}_{\rho-\mu}(x_1^{-1}, \dots, x_n^{-1}) \cdot \prod_{1 \leq i < j \leq n} (x_i - x_j) \cdot (1 + x_{n+1}^{-N+n+1} \dots x_{N-1}^{-1} R)$$

modulo terms with degree > 0 in variables x_{n+1}, \dots, x_N . Since x_{n+1}, \dots, x_N does not appear in \mathfrak{S}_w and $x_{n+1}^{-N+n+1} \dots x_{N-1}^{-1} R$ does not have a constant term, this shows

$$\langle \mathfrak{S}_w, \mathfrak{S}_{w_0 v}(x_1^{-1}, \dots, x_N^{-1}) \prod_{1 \leq i < j \leq N} (x_i - x_j) \rangle = \langle \mathfrak{S}_w, \mathfrak{S}_{\rho-\mu}(x_1^{-1}, \dots, x_n^{-1}) \prod_{1 \leq i < j \leq n} (x_i - x_j) \rangle.$$

This, together with (*), finishes the proof of Lemma 7.4. \square

Let us come back to the proof of Lemma 7.2. Essentially this is a ‘‘Cauchy formula’’ for the dual bases $\{\mathfrak{S}_\lambda\}$ and $\{\mathfrak{S}_{\rho-\mu}(x_1^{-1}, \dots, x_n^{-1}) \prod (x_i - x_j)\}$ appeared in Lemma 7.4, but since we are dealing with an infinite-dimensional space a careful justification is needed. Let $c_{\alpha\beta}$ be the coefficient of $x^\alpha y^\beta$ in $\sum_{\kappa \in \mathbb{Z}^n} \mathfrak{S}_{\rho-\kappa}(x) \mathfrak{S}_\kappa(y)$. We observe that if $c_{\rho-\mu, \nu} \neq 0$, then there exists some κ such that $\rho - \mu \geq \rho - \kappa$ and $\nu \geq \kappa$, and so $\nu \geq \kappa \geq \mu$. Thus $c_{\rho-\mu, \nu} = 0$ for $\nu \not\geq \mu$. Using this as the base case, if we show $\sum_{g \in S_n} \text{sgn}(g) c_{\rho-\mu, \nu - \rho + g\rho} = \delta_{\mu\nu}$, then we can show $c_{\rho-\mu, \nu} = \dim \mathcal{U}(\mathfrak{n}^+)_{\nu-\mu}$ by induction on ν since $\sum_{\kappa} \dim \mathcal{U}(\mathfrak{n}^+)_{\kappa} x^\kappa = \prod_{i < j} (1 - x_i x_j^{-1})^{-1}$ and $\prod_{i < j} (1 - x_i x_j^{-1}) = \sum_{g \in S_n} \text{sgn}(g) x^{\rho - g\rho}$. We show below the equivalent claim $\sum_{g \in S_n} \text{sgn}(g) c_{\alpha, \beta + g\rho} = \delta_{\alpha, -\beta}$.

Since $c_{\alpha, \beta + g\rho} = c_{\alpha + k\mathbf{1}, \beta + g\rho - k\mathbf{1}}$, we may assume that $-\beta \in \mathbb{Z}_{\geq 0}^n$. We may further assume, by replacing α and β by $\alpha + k\mathbf{1}$ and $\beta - k\mathbf{1}$ for a sufficiently large k , that if $\kappa \in \mathbb{Z}^n$ satisfies $\alpha \succeq \kappa \succeq -\beta + \rho - g\rho$ for some $g \in S_n$ then $\kappa \in \mathbb{Z}_{\geq 0}^n$ (this is possible by the remark at the end of Section 3). Also it is sufficient to consider the case $|\alpha| = -|\beta|$. Let $d = |\alpha|$. Let V be the space of all (ordinary) polynomials in x_1, \dots, x_n which are homogeneous of degree d . Equip V with a bilinear form $\langle x^\sigma, x^\tau \rangle = \delta_{\sigma\tau}$. Then by Lemma 7.4 the bases $\{\mathfrak{S}_\kappa : \kappa \in \mathbb{Z}_{\geq 0}^n, |\kappa| = d\}$ and $\{[\mathfrak{S}_{\rho-\kappa}(x_1^{-1}, \dots, x_n^{-1}) \prod_{1 \leq i < j \leq n} (x_i - x_j)] : \kappa \in \mathbb{Z}_{\geq 0}^n, |\kappa| = d\}$ of V are dual of each other; here for $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $[f]$ is the sum of all terms in f which do not contain any negative powers of x_1, \dots, x_n . Thus we have

$$\begin{aligned} \sum_{\substack{\gamma \in \mathbb{Z}_{\geq 0}^n \\ |\gamma| = d}} x^\gamma y^\gamma &\equiv \sum_{\substack{\kappa \in \mathbb{Z}_{\geq 0}^n \\ |\kappa| = d}} \mathfrak{S}_\kappa(x_1, \dots, x_n) \mathfrak{S}_{\rho-\kappa}(y_1^{-1}, \dots, y_n^{-1}) \prod_{1 \leq i < j \leq n} (y_i - y_j) \\ &= \left(\sum_{\substack{\kappa \in \mathbb{Z}_{\geq 0}^n \\ |\kappa| = d}} \mathfrak{S}_\kappa(x_1, \dots, x_n) \mathfrak{S}_{\rho-\kappa}(y_1^{-1}, \dots, y_n^{-1}) \right) \left(\sum_{g \in S_n} \text{sgn}(g) y^{g\rho} \right) \cdots (*) \end{aligned}$$

modulo terms containing some negative powers of some y_i (note that for any finite-dimensional vector space V , the sum $\sum \phi_i \otimes \phi_i^* \in V \otimes V^*$ does not depend on the choice of dual bases $\{\phi_i\} \subset V, \{\phi_i^*\} \subset V^*$). Since $-\beta \in \mathbb{Z}_{\geq 0}^n$, the coefficient of $x^\alpha y^{-\beta}$ is equal for both side. The coefficient for the LHS is $\delta_{\alpha, -\beta}$. Moreover, if $\kappa \in \mathbb{Z}^n$ and $\mathfrak{S}_\kappa(x_1, \dots, x_n) \mathfrak{S}_{\rho-\kappa}(y_1^{-1}, \dots, y_n^{-1})$ contains some monomial of the form $x^\alpha y^{-\beta - g\rho}$ ($g \in S_n$) with nonzero coefficients, then such κ indeed satisfies $\kappa \in \mathbb{Z}_{\geq 0}^n$ and thus appears in the first sum in $(*)$ above, since such κ must satisfy $\alpha \succeq \kappa$ and $\beta + g\rho \succeq \rho - \kappa$. So the coefficient of $x^\alpha y^{-\beta}$ in the RHS is the same as the coefficient of $x^\alpha y^{-\beta}$ in $\left(\sum_{\substack{\kappa \in \mathbb{Z}^n \\ |\kappa| = d}} \mathfrak{S}_\kappa(x_1, \dots, x_n) \mathfrak{S}_{\rho-\kappa}(y_1^{-1}, \dots, y_n^{-1}) \right) \left(\sum_{g \in S_n} \text{sgn}(g) y^{g\rho} \right)$. Since this coefficient is $\sum_{g \in S_n} \text{sgn}(g) c_{\alpha, \beta + g\rho}$ we are done. \square

Remark 7.5. This proof, together with some results from the previous section, in fact shows that $\mathcal{C}_{\leq \lambda}$ can be equipped with a structure of highest-weight category ([1]) whose standards and costandards are \mathcal{S}_μ ($\mu \leq \lambda$) and $\mathcal{S}_{\rho-\nu}^* \otimes K_\rho$ ($\nu \leq \lambda$) respectively. The results in the next section is then a standard argument in the theory of highest-weight categories. I would like to thank Katsuyuki Naoi for giving the author this information.

From Lemma 7.1, we obtain the following corollary. This can be seen as an analog of “Strong form of Polo’s theorem” ([15, Theorem 3.2.2]) for KP modules.

Corollary 7.6. *For $\lambda \in \mathbb{Z}^n$, $\mu, \nu \leq \lambda$ and $i \geq 1$, $\text{Ext}_{\leq \lambda}^i(\mathcal{S}_\mu, \mathcal{S}_{\rho-\nu}^* \otimes K_\rho) = 0$.*

Proof. By Lemma 7.1, it suffices to prove $\text{Ext}_{\leq \max\{\mu, \nu\}}^i(\mathcal{S}_\mu, \mathcal{S}_{\rho-\nu}^* \otimes K_\rho) = 0$. If $\mu \geq \nu$, this follows from the projectivity of $\mathcal{S}_\mu \in \mathcal{C}_{\leq \mu}$ since $\mathcal{S}_{\rho-\nu}^* \otimes K_\rho \in \mathcal{C}_{\leq \mu}$. Otherwise it follows from the injectivity of $\mathcal{S}_{\rho-\nu}^* \otimes K_\rho \in \mathcal{C}_{\leq \nu}$ since $\mathcal{S}_\mu \in \mathcal{C}_{\leq \nu}$. \square

8 Existence of KP filtrations

Using the results obtained so far, we can obtain a criterion for a module to have a KP filtration, using the similar argument from [15, §3]. Hereafter, $\text{Ext}^i(M, N)$ means $\text{Ext}_{\leq \lambda}^i(M, N)$ for a suitable λ (by Lemma 7.1 this does not depend on the choice of λ).

Theorem 8.1. *Let $\lambda \in \mathbb{Z}^n$, $M \in \mathcal{C}_{\leq \lambda}$ and assume that $\text{Ext}^1(M, \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$ for all $\mu \leq \lambda$. Then M has a filtration such that each of its subquotients is isomorphic to some \mathcal{S}_ν ($\nu \leq \lambda$).*

Note that the converse obviously holds since $\text{Ext}^1(\mathcal{S}_\nu, \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$.

Proof. Let $\{\nu : \nu \leq \lambda\} = \{\nu^{(1)} < \nu^{(2)} < \dots < \nu^{(r)}\}$. Let M_i be the largest quotient of M whose weights are in $\{\nu^{(1)}, \dots, \nu^{(i)}\}$ (so $M_0 = 0$ and $M_r = M$). By definition, we have a natural surjection $M_i \twoheadrightarrow M_j$ for $i > j$. We show that $\text{Ker}(M_i \twoheadrightarrow M_{i-1})$ is a direct sum of some copies of \mathcal{S}_ν by the induction on i . This will show that $M = M_r \twoheadrightarrow M_{r-1} \twoheadrightarrow \dots \twoheadrightarrow M_0 = 0$ gives a quotient filtration with desired property.

Fix i and let $\nu = \nu^{(i)}$. It is sufficient to show that $\text{Ker}(M_i \twoheadrightarrow M_{i-1})$ is the projective cover of its ν -weight space $\text{Ker}(M_i \twoheadrightarrow M_{i-1})_\nu$ in $\mathcal{C}_{\leq \nu}$, since \mathcal{S}_ν is the projective cover of K_ν in $\mathcal{C}_{\leq \nu}$. Since $\text{Ker}(M_i \twoheadrightarrow M_{i-1})$ is generated by $\text{Ker}(M_i \twoheadrightarrow M_{i-1})_\nu$, it suffices to show that $\text{Ker}(M_i \twoheadrightarrow M_{i-1})$ is projective in $\mathcal{C}_{\leq \nu}$, that is, $\text{Ext}^1(\text{Ker}(M_i \twoheadrightarrow M_{i-1}), K_\mu) = 0$ for all $\mu \leq \nu$.

Let $\mu \leq \nu$. We have an exact sequence $\text{Ext}^1(M, \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) \rightarrow \text{Ext}^1(\text{Ker}(M \twoheadrightarrow M_{i-1}), \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) \rightarrow \text{Ext}^2(M_{i-1}, \mathcal{S}_{\rho-\mu}^* \otimes K_\rho)$. Here $\text{Ext}^1(M, \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$ by the hypothesis. Moreover, $\text{Ext}^2(M_{i-1}, \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$ by Corollary 7.6, since M_{i-1} has a filtration by modules \mathcal{S}_κ ($\kappa < \nu$) by the induction hypothesis. Therefore $\text{Ext}^1(\text{Ker}(M \twoheadrightarrow M_{i-1}), \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$.

We have an exact sequence $\text{Hom}(\text{Ker}(M \twoheadrightarrow M_i), \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) \rightarrow \text{Ext}^1(\text{Ker}(M_i \twoheadrightarrow M_{i-1}), \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) \rightarrow \text{Ext}^1(\text{Ker}(M \twoheadrightarrow M_{i-1}), \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$. But here $\text{Hom}(\text{Ker}(M \twoheadrightarrow M_i), \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$, since the weights of $\mathcal{S}_{\rho-\mu}^* \otimes K_\rho$ are all less than or equal to μ and therefore $\leq \nu$, while $\text{Ker}(M \twoheadrightarrow M_i)$ is generated by the elements whose weights are $> \nu$. Therefore $\text{Ext}^1(\text{Ker}(M_i \twoheadrightarrow M_{i-1}), \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$.

We have an exact sequence $\text{Hom}(\text{Ker}(M_i \twoheadrightarrow M_{i-1}), (\mathcal{S}_{\rho-\mu}^* \otimes K_\rho)/K_\mu) \rightarrow \text{Ext}^1(\text{Ker}(M_i \twoheadrightarrow M_{i-1}), K_\mu) \rightarrow \text{Ext}^1(\text{Ker}(M_i \twoheadrightarrow M_{i-1}), \mathcal{S}_{\rho-\mu}^* \otimes K_\rho) = 0$. But since the weights of $(\mathcal{S}_{\rho-\mu}^* \otimes K_\rho)/K_\mu$ are strictly less than μ and thus $< \nu$ while $\text{Ker}(M_i \twoheadrightarrow M_{i-1})$ is generated by its ν -weight space, $\text{Hom}(\text{Ker}(M_i \twoheadrightarrow M_{i-1}), (\mathcal{S}_{\rho-\mu}^* \otimes K_\rho)/K_\mu) = 0$. So $\text{Ext}^1(\text{Ker}(M_i \twoheadrightarrow M_{i-1}), K_\mu) = 0$ and we are done. \square

Another criterion for filtration can be also derived:

Theorem 8.2. *Let $\lambda \in \mathbb{Z}^n$ and $M \in \mathcal{C}_{\leq \lambda}$. Then $\text{ch}(M) \leq \sum_{\nu \leq \lambda} \dim_K(\text{Hom}(M, \mathcal{S}_{\rho-\nu}^* \otimes K_\rho)) \mathfrak{S}_\nu$, and the equality holds if and only if M has a filtration such that each of its subquotients is isomorphic to some \mathcal{S}_ν ($\nu \leq \lambda$). Here $\sum a_\alpha x^\alpha \leq \sum b_\alpha x^\alpha$ is defined as $a_\alpha \leq b_\alpha$ ($\forall \alpha$).*

Proof. Let $\{\nu : \nu \leq \lambda\} = \{\nu^{(1)} < \nu^{(2)} < \dots < \nu^{(r)}\}$. By the proof of Theorem 8.1, M has a desired filtration if and only if $\text{Ker}(M_i \twoheadrightarrow M_{i-1})$ is a direct sum

of some copies of $\mathcal{S}_{\nu^{(i)}}$, where M_i is the largest quotient of M whose weight are in $\{\nu^{(1)}, \dots, \nu^{(i)}\}$.

We have $\text{ch}(M) = \sum_i \text{ch}(\text{Ker}(M_i \twoheadrightarrow M_{i-1}))$. Since $\text{Ker}(M_i \twoheadrightarrow M_{i-1})$ is generated by its $\nu^{(i)}$ -weight space $(M_i)_{\nu^{(i)}}$, if we let d_i denote the dimension of this weight space, we have a surjection from $(P_{\nu^{(i)}}^{\leq \nu^{(i)}})^{\oplus d_i}$ to $\text{Ker}(M_i \twoheadrightarrow M_{i-1})$. We have seen in Proposition 6.4 that $P_{\nu^{(i)}}^{\leq \nu^{(i)}} \cong \mathcal{S}_{\nu^{(i)}}$. Thus $\text{ch}(M) \leq \sum_i \dim((M_i)_{\nu^{(i)}}) \mathfrak{S}_{\nu^{(i)}}$ and the equality holds when and only when each kernel is a direct sum of some copies of $\mathcal{S}_{\nu^{(i)}}$, i.e. M has a desired filtration.

For each i , we have $\text{Hom}(M_i, \mathcal{S}_{\rho-\nu}^* \otimes K_\rho) \cong \text{Hom}(\mathcal{S}_{\rho-\nu}, M_i^* \otimes K_\rho) \cong (M_i^* \otimes K_\rho)_{\rho-\nu} \cong ((M_i)_\nu)^*$ where $\nu = \nu^{(i)}$, since $M_i^* \otimes K_\rho \in \mathcal{C}_{\{\rho-\mu: \mu \leq \nu\}} = \mathcal{C}_{\leq' \rho-\nu}$ and $\mathcal{S}_{\rho-\nu}$ is the projective cover of $K_{\rho-\nu}$ in this category. Thus the theorem follows. \square

9 Questions

Question 9.1. For $\lambda, \mu \in \mathbb{Z}^n$, does $\mathcal{S}_\lambda \otimes \mathcal{S}_\mu$ have a KP filtration?

By the criteria obtained above, this question is equivalent to ask:

- whether $\text{Ext}^1(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu, \mathcal{S}_\nu^* \otimes K_\rho) = 0$ or not, or
- whether the dimension of $\text{Hom}(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu, \mathcal{S}_{\rho-\nu}^* \otimes K_\rho)$ is equal to the coefficient of \mathfrak{S}_ν in the expansion of $\mathfrak{S}_\lambda \mathfrak{S}_\mu$ into a linear combination of Schubert polynomials.

Question 9.2. Let s_σ denote the Schur functor corresponding to a partition σ and let $\lambda \in \mathbb{Z}^n$. Then, does $s_\sigma(\mathcal{S}_\lambda)$ have a KP filtration?

As explained in the introduction, positive answer for this question implies that the “plethysm” $s_\sigma[\mathfrak{S}_\lambda]$ is a positive sum of Schubert polynomials.

We note the following connection between these two problems.

Proposition 9.3. Suppose that the answer to Question 9.1 is yes. Then the answer to Question 9.2 is yes.

Proof. By iteratively using 9.1, we see that $\mathcal{S}_{\lambda^{(1)}} \otimes \dots \otimes \mathcal{S}_{\lambda^{(r)}}$ has a KP filtration for any $\lambda^{(1)}, \dots, \lambda^{(r)} \in \mathbb{Z}^n$. Especially, $(\mathcal{S}_\lambda)^{\otimes k}$ has a KP filtration for any λ and k . Therefore $\text{Ext}^1((\mathcal{S}_\lambda)^{\otimes k}, \mathcal{S}_\nu^* \otimes K_\rho) = 0$ for any ν . Since $s_\sigma(\mathcal{S}_\lambda)$ is a direct sum factor of $(\mathcal{S}_\lambda)^{\otimes |\sigma|}$, $\text{Ext}^1(s_\sigma(\mathcal{S}_\lambda), \mathcal{S}_\nu^* \otimes K_\rho) = 0$. Thus $s_\sigma(\mathcal{S}_\lambda)$ has a KP filtration by Theorem 8.1. \square

Note. in a subsequent work the author gave positive answers to both of the questions above: see [16].

References

- [1] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.
- [2] S. Fomin, C. Greene, V. Reiner, and M. Shimozono. Balanced labellings and Schubert polynomials. *Eur. J. Comb.*, 18(4):373–389, 1997.

- [3] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer-Verlag New York, 1972.
- [4] A. Joseph. On the Demazure character formula. *Ann. Sci. École Norm. Sup. (4)*, 18(3):389–419, 1985.
- [5] W. Kraśkiewicz and P. Pragacz. Foncteurs de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(9):209–211, 1987.
- [6] W. Kraśkiewicz and P. Pragacz. Schubert functors and Schubert polynomials. *Eur. J. Comb.*, 25(8):1327–1344, 2004.
- [7] A. Lascoux. *Symmetric Functions and Combinatorial Operators on Polynomials*. C.B.M.S. Reg. Conf. Ser. Maths. 99, 2003.
- [8] A. Lascoux and M.-P. Schützenberger. Tableaux and noncommutative Schubert polynomials. *Funct. Anal. Appl.*, 23:63–64, 1989.
- [9] I. G. Macdonald. *Notes on Schubert Polynomials*. LACIM, Université du Québec à Montréal, 1991.
- [10] I. G. Macdonald. *Symmetric Functions and Hall Polynomials, second edition*. Oxford University Press, 1999.
- [11] P. Magyar. Four new formulas for Schubert polynomials. <http://math.msu.edu/~magyar/papers/FourFormulas.pdf>.
- [12] P. Polo. Variétés de Schubert et excellentes filtrations. *Astérisque*, (173-174):10–11, 281–311, 1989. Orbites unipotentes et représentations, III.
- [13] A. Postnikov and R. P. Stanley. Chains in the Bruhat order. *J. Algebraic Combin.*, 29:133–74, 2009.
- [14] W. van der Kallen. Longest weight vectors and excellent filtrations. *Math. Z.*, 201(1):19–31, 1989.
- [15] W. van der Kallen. *Lectures on Frobenius Splittings and B-modules*. Springer, 1993.
- [16] M. Watanabe. Tensor product of Kraśkiewicz and Pragacz’s modules. preprint, [arXiv:1410.7981v1](https://arxiv.org/abs/1410.7981), 2014.